Atypical dynamics of materials with periodic microstructure and local resonance

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Summary

This work investigates the dynamic behavior of periodic unbraced frame structures made up of interconnected beams. Two types of microstructures are especially studied: non-orthogonal unbraced frame and honeycombs.The microstructure being much stiffer in compression than in shear, a great variety of behaviors can occur. Assuming the condition of scale separation is respected, the dynamical behaviors at the leading order are approached by the homogenisation method of periodic discrete media. In the studied ranges, the local elements behave ever in quasi-statics, ever in dynamics. For studied materials, the elastic law are given in function of the elements properties. These laws correspond to upgraded materials as double gradient media or meta-material. To illustrate their atypical properties, propagations of 'shear' and 'compression' waves are studied. In the presence of the local resonance, the form of the equations is unchanged but the mass depends on the frequency and, as a result, frequency bandgaps appear.

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1. Introduction

Two considerations may explain the great number of studies devoted to the dynamic properties of periodic reticulated (or cellular) structures, namely structures obtained by repeating a unit cell made up of interconnected beams (or plates). First they are frequently encountered: in sandwich panels, stiffened plates, truss beams used in aerospace and marine structures. Second, the periodic materials can generate complex behavior (Cauchy or generalized media, meta-material).

This work analyzes the propagation of plane waves in two-dimensional periodic materials (Fig 1), constituted of elements oriented in two or three directions. These elements being more flexible in bending than in compression, the macroscopic properties of such materials can present a great variability in function of the direction and of the frequency of the solicitation.

The homogenization method of periodic discrete media (HPDM) is used. This method has already given interesting results on the dynamic behavior of frame structures [5], [6]. Its advantages are:

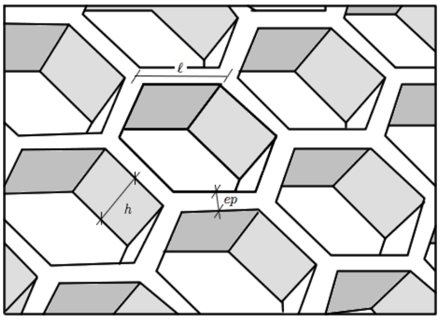

Figure 1. Example of structures

(1) The equivalent continuum is derived rigorously from the properties of the cell. The only assumption is the scale separation, i.e. two scales with very different characteristic lengths can be defined : the macroscopic (or global) scale is given by the wave propagation and the microscopic (or local) scale by the size of the cell.
(2) The method is completely analytic. This provides a clear understanding of the mechanisms governing the behavior and of the role of each parameter. Such a knowledge is desirable for the design of new materials with prescribed properties.
(3) The global behavior being identified, it is always possible to come back to the local scale to determine

the deformations and the efforts in the elements.
(4) Superior orders of the expansions are obtained relatively easily. This is particularly interesting for the frame structures because the shear and the compression stiffnesses do not have the same order of magnitude. Since the method of multiple parameters and scale changes is generally limited to the leading order, it misses the shear properties and the coerciveness of the macroscopic description is lost [1].
The implementation of the HPDM method is realized in two steps [4] [3] : the discretization of the momentum balance and the homogenization process itself. As in [5], [6] or [7], the HPDM method is coupled with the scaling of all the parameters in order to correctly take into account the physics of the problem.

Section 2 describes the studied structures and section 3 the principles of the HPDM method. Then the equivalent description obtained in the absence of local resonance for the inclined lattice is presented in Section 4 and the wave propagation is analyzed at two frequency ranges. In Section 5, the results on honeycomb are presented before a short conclusion.

2. Studied structures and framework of analysis

The studied structures (Fig. 2) are infinite and periodic in the plane (x, y). In the first case named *inclined lattice*, the fundamental cell is constituted by two elements of length ℓ_f and ℓ_w whose inclination angles differ of φ, and in the second case named *honeycomb*, the fundamental cell is constituted by three identical elements inclined of $\pi/3$ the ones by regards to the others. Elements are beams or plates behaving as Euler beams in the plane (x, y). They are linked by perfectly stiff and massless nodes of coordinates $\vec{x}_{np} = (x_{np}, y_{np})$. Moreover, all elements have similar material and geometric properties.

The study is conducted within the framework of the small strain theory, the linear elasticity and in harmonic regime.

3. Homogenisation of periodic discrete media

3.1. Discretization of the dynamic balance

The discretization step consists in integrating the dynamic balance of the beams, taking the unknown displacements and rotations $\vec{u} = (u, v, \theta)$ at their extremities as boundary conditions (Fig.3). The efforts (forces and moment) applied by an element on its extremities are expressed explicitly as functions of the nodal kinematic variables.

$$\begin{array}{llr} \text{Compression} & & N = f(\vec{u}^B, \vec{u}^E, \ell/\lambda_c) \\ \text{Bending} & T \text{ or } & M = f(\vec{u}^B, \vec{u}^E, \ell/\lambda_b) \end{array} \quad (1)$$

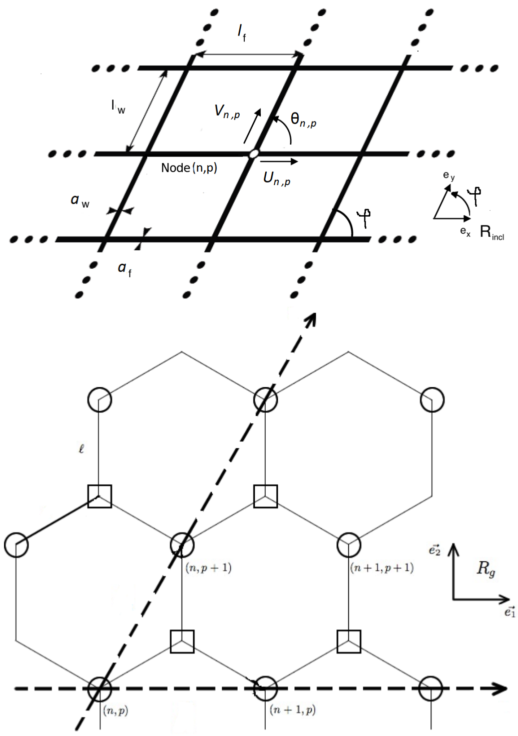

Figure 2. Notations (above : inclined lattice - below : honeycomb)

where λ_c and λ_b are the wavelengths in the local elements at the circular frequency ω ($\lambda_c \gg \lambda_b$).

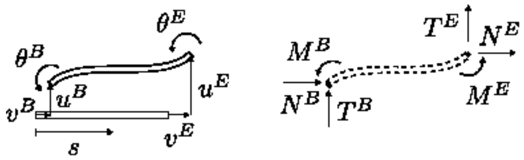

Figure 3. Notation (element)

The local dynamic balance of each element being satisfied, it remains to write the balance of the nodes. It consists in adding the efforts applied by the elements connected to the same node. The geometry of the structure is also explicitly taken into account.

This process reduces the balance of the whole structure to the balance of the set of nodes without any assumption.

3.2. Scale separation and consequences

The principles of homogenization are now used to derive the differential equations describing the behavior of the equivalent continuum. The key assumption is scale separation. This means that the characteristic length L of the deformation of the structure under vibrations is assumed to be much greater than the characteristic length ℓ_c of the basic cell. Thus, the scale ratio $\epsilon = \ell_c/L$ is a small parameter ($1 \gg \epsilon$). Under scale separation condition, the nodal motions vary slowly from one node to the next. Therefore, the nodal variables ($U^{(n,p)}$, $V^{(n,p)}$, $\theta^{(n,p)}$) is described as the discrete values of continuous functions of space variables x and y, written as asymptotic form:

$$X^{(n,p)} = X(\epsilon, \vec{x}) = X^0(\vec{x}) + \epsilon\, X^1(\vec{x}) + \dots \quad (2)$$

In the sequel, the physically observable variables of a given order are written with a tilde: $\tilde{X}^j(\vec{x}) = \epsilon^j X^j(\vec{x})$. The size of the cell being small (compared to L), the motions of the neighboring nodes of node (n,p) are obtained by Taylor's series, what introduces the macroscopic derivatives:

$$\begin{aligned} &X^{(n\pm1,p\pm1)} = X^0(\vec{x}) + \dots \\ &\epsilon\left(X^1(\vec{x}) \pm \ell_x^* L\, \partial_x X^0(\vec{x}) \pm \ell_y^* L\, \partial_y X^0(\vec{x})\right) + \dots \end{aligned}$$

Concerning the efforts, two situations are considered : (1) at the excitation frequency, the elements are in quasi-static regime (i.e $\lambda_c \gg \lambda_b \gg \ell_c$) : the expressions of all efforts (N, T, M) can be developed in Taylor's series, (2) at the excitation frequency, the elements are in dynamic regime for the bending (i.e $\lambda_c \gg \lambda_b \simeq \ell_c$) : only the expressions of compression force (N) are developed in Taylor's series, the others (T and M) are conserved unchanged.

3.3. Normalization

Normalization consists of scaling the physical parameters (the properties of the elements and the circular frequency) according to ϵ. It ensures that each mechanical effect appears at the same order whatever the value of ϵ. Thus, the same physics is kept at the limit $\epsilon \to 0$, which represents the homogenized model.

The choice of the properties of the elements determines the stiffness contrast and then the possible mechanisms in the structure. Here, for the two structures , the elements have similar geometrical and material properties, and a thickness to length ratio of order $\epsilon^{1/2}$ or ϵ.

As regards the circular frequency, its scaling enables to explore the different dynamic regime of the global structure : the shear waves classically appears at lower frequencies than the compression waves. So, the frequency order (in ϵ) will be precised in each situation.

3.4. Continuous description

Finally, the expansions in powers of ϵ [§(3.2)] and the scaling of the parameters [§(3.3)] are introduced in the balance equation of the nodes. The relations obtained being valid for any small enough ϵ, the orders can be separated. This leads to a set of differential equations for each order, which can be solved in increasing order.

The homogenized model is given by the leading order, which corresponds to the limit when ϵ approaches zero. However, in a real structure, the macroscopic length L and the microscopic length ℓ_c are finite and the physical scale ratio $\tilde{\epsilon}$ is necessarily a finite quantity. Consequently, the kinematic variables of order 0 ($\tilde{U}^0$, $\tilde{V}^0$, and $\tilde{\theta}^0$) are an approximation of the real motion (the accuracy of which depends on the order of magnitude of $\tilde{\epsilon}$). The terms of superior orders are correctors which improve the accuracy of the macroscopic description by taking into account phenomena of lesser importance.

4. Study of inclined lattice

To simplify the analysis, the study of the inclined lattice (Fig.2) is realised in the global inclined coordinate system whose axes are parallel to the elements of the lattice. The following sections focus on the leading order. First the equivalent continuum is characterized and then the wave propagation is studied.

In order to simplify the equations, some macroscopic parameters are defined. They are integrated over the depth of the elements so that they do not have the usual units :
- $M_s = M_w + M_f$: mass per unit surface (kg/m^2),
- $E_x = E_f A_f/\ell_w \sin(\varphi)$ and $E_y = E_w A_w/\ell_f \sin(\varphi)$: elastic modulus in the x and y-direction (N/m),
- $G_w = \frac{12 E_w I_w}{\ell_w^2 \ell_f \sin(\varphi)}$ and $G_f = 12\, \frac{12 E_f I_f}{\ell_w \ell_f^2 \sin(\varphi)}$: contribution of the walls (w) and the floors (f) to the shear modulus (N/m),
- $G^{-1} = G_w^{-1} + G_f^{-1}$: shear modulus (N/m)

4.1. Continuous description

For this structure, the equilibrium of the node (n,p) depends of the motions of the four neighboring nodes $n \pm 1, p$ and $n, p \pm 1$, so :

$$\begin{aligned} &\vec{F}^E(\vec{U}^{n-1,p}, \vec{U}^{n,p}) - \vec{F}^B(\vec{U}^{n,p}, \vec{U}^{n+1,p}) + \\ &\vec{F}^E(\vec{U}^{n,p-1}, \vec{U}^{n,p}) - \vec{F}^B(\vec{U}^{n,p}, \vec{U}^{n,p+1}) = 0 \end{aligned} \quad (3)$$

In this section and the next present the behavior of the lattice at the lowest circular frequencies giving a dynamic description: $\omega = O(\epsilon\, \omega_r)$. In this case, the implementation of the HPDM method provides from Eq.3 the following continuous description:

$$\left\{\begin{aligned} &E_x\, \partial_{xx}(\tilde{U}^0 + c_\varphi \tilde{V}^0) = 0 && (x\,0) \\ &E_x\, \partial_{xx}(\tilde{U}^2 + c_\varphi \tilde{V}^2) + && (x\,2) \\ &G\,(\partial_y - c_\varphi \partial_x)(\partial_y \tilde{U}^0 + \partial_x \tilde{V}^0) + \\ &(E_x \ell_f^2/12)\, \partial_{xxxx}(\tilde{U}^0 + c_\varphi \tilde{V}^0) + M_s\, \omega^2\, \tilde{U}^0 = 0 \end{aligned}\right.$$

$$\left\{\begin{aligned} &E_y\, \partial_{yy}(c_\varphi \tilde{U}^0 + \tilde{V}^0) = 0 && (y\,0) \\ &E_y\, \partial_{yy}(c_\varphi \tilde{U}^2 + \tilde{V}^2) + && (y\,2) \\ &G\,(\partial_x - c_\varphi \partial_y)(\partial_y \tilde{U}^0 + \partial_x \tilde{V}^0) + \\ &(E_y \ell_w^2/12)\, \partial_{yyyy}(c_\varphi \tilde{U}^0 + \tilde{V}^0) + M_s\, \omega^2\, \tilde{V}^0 = 0 \end{aligned}\right.$$

$$\left\{ G_w\left(\tilde{\theta}^0 + s_\varphi \partial_y \tilde{U}^0\right) + G_f\left(\tilde{\theta}^0 - s_\varphi \partial_x \tilde{V}^0\right) = 0 \right.$$

where $s_\varphi = \sin(\varphi)$ and $c_\varphi = \cos(\varphi)$.

The main feature of the continuous description is its extreme anisotropy due to the large difference in magnitude of the moduli E_x, E_y (that appears in the first order equation $(x\ 0)$ and $(y\ 0)$) and G (that appears only at the second order). Because of the quasi-static state at the local scale, the moduli only depend on the *elastostatic* properties of the frame elements. The two elastic moduli, E_x and E_y, are related to the tension-compression rigidity of the floors and to the one of the walls respectively. On the contrary, the shear mechanism results from the bending of the walls and the floors connected in series. Since beams are far less stiff in bending, the shear modulus G is much less than the elastic moduli:

$$\frac{G}{E_x} = O(\epsilon^2) \qquad \frac{G}{E_y} = O(\epsilon^2)$$

This is the reason why it is necessary to calculate equations up to order 2.

The macroscopic behavior is completely described by Eqs. $(x\,0)$, $(x\,2)$, $(y\,0)$, and $(y\,2)$ which do not contain $\tilde{\theta}^0$. The node rotation has the status of a "hidden" variable. However, to come back to the local scale and to determine the forces and the displacements in the frame elements, it is necessary to calculate $\tilde{\theta}^0$ with Eq.(4) describing the inner equilibrium of the basic frame.

$$\tilde{\theta}^0 = \frac{\sin(\varphi)}{G_w + G_f}\left(G_f\,\partial_x \tilde{V}^0 - G_w\,\partial_y \tilde{U}^0\right) \tag{4}$$

Finally, note that the previous description of the macroscopic medium established for circular frequencies such that $\omega = O(\epsilon\,\omega_r)$ remains valid as long as the frame elements are not in resonance in bending. In particular, it applies to statics.

4.2. Shear waves

The wave propagation in the medium is now analyzed. Since every wave can be expressed as a superposition of plane waves, the study focuses on this kind of waves and the displacement field is sought in the following way (remember that the time dependence $\exp(i\omega t)$ is systematically omitted):

$$\vec{U}(\epsilon, x) = (\vec{u}^0 + \epsilon\,\vec{u}^1 + \epsilon^2\vec{u}^2 + \ldots)\exp^{-ik(\alpha)\,\vec{n}_\alpha\cdot\vec{x}} \tag{5}$$

Expression (5) is introduced in Eqs. $(x\,0)$ and $(y\,0)$:

$$-E_x\,k^2(\alpha)\,\cos^2(\alpha)\,(\tilde{u}^0 + \cos(\varphi)\tilde{v}^0) = 0 \qquad (x\,0)$$
$$-E_y\,k^2(\alpha)\,\sin^2(\alpha)\,(\cos(\varphi)\tilde{u}^0 + \tilde{v}^0) = 0 \qquad (y\,0)$$

For $\cos(\alpha) \neq 0$ and $\sin(\alpha) \neq 0$, the only solution is $\vec{u}^0 = \vec{0}$. At this frequency range, only two directions of propagation are possible.

For $\alpha = 0$, Eq. $(x\,0)$ implies $\tilde{u}^0 + \cos(\varphi)\tilde{v}^0 = 0$. Then the second order equation $(y\,2)$ gives:

$$\left(-G\,k^2(0) + M_s\,\omega^2\right)\tilde{v}^0 = 0 \tag{6}$$

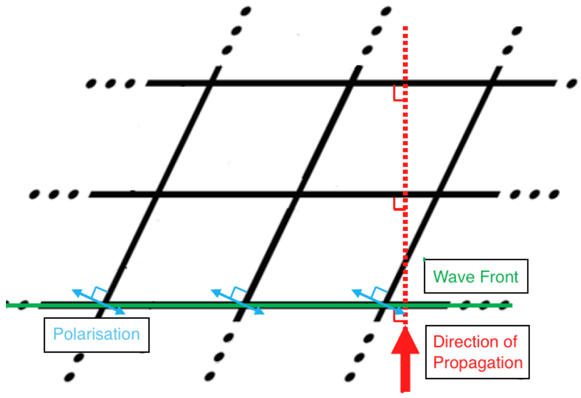

Figure 4. Shear wave traveling in the $\pi/2$ direction ($\alpha = pi/2$)

For $\alpha = \pi/2$, the results are similar, but the roles of $\tilde{u}^0$ and $\tilde{v}^0$ are reversed. To correctly understand the results, they have to be transposed in a orthogonal coordinates system. This is illustrated on the figure 4. The real velocity is also obtained by this way and is found constant in the two directions:

$$c = \sin(\varphi)\sqrt{\frac{G}{M_s}} \tag{7}$$

To sum up, at low frequencies, waves can only propagate in two directions because of the anisotropy. The speed depends on the shear modulus G and the mass M_s as in a classical elastic medium. The expression of G (given in Section 4) shows that these waves are generated by the local bending of the elements.

4.3. Compression waves

The circular frequency ω is now increased up to $O(\omega_r)$ in order to investigate the behavior of the medium when the inertia forces balance the tension-compression forces. At this frequency, in function of thickness, the elements can be in quasi-static regime or in dynamics (resonance in bending). The static case is treated first and then the inner resonance.

In statics also, with the increase of the frequency, the inertial terms appear in the first order equations:

$$E_x\,\partial_{xx}(\tilde{U}^0 + c_\varphi\tilde{V}^0) + M_s\,\omega^2\,\tilde{U}^0 = 0 \qquad (x\ 0)'$$

$$E_y\,\partial_{yy}(c_\varphi\tilde{U}^0 + \tilde{V}^0) + M_s\,\omega^2\,\tilde{V}^0 = 0 \qquad (y\ 0)'$$

The analysis of the wave propagation is carried out using the same method as in Section 4.2. Expression (5) of the displacement field is introduced in Eqs. $(x\,0)'$ and $(y\,0)'$:

$$E_x\,k^2(\alpha)\,\cos^2(\alpha)\,(\tilde{u}^0 + c_\varphi\tilde{v}^0) = M_s\,\omega^2\,\tilde{u}^0 \quad (x\ 0)'$$

$$E_y\,k^2(\alpha)\,\sin^2(\alpha)\,(c_\varphi\tilde{u}^0 + \tilde{v}^0) = M_s\,\omega^2\,\tilde{v}^0 \quad (y\ 0)'$$

To search non-zero solutions implies to nullify the determinant of the system. This gives, whatever the value of α is, two solutions $k_1(\alpha)$ and $k_2(\alpha)$. It means the propagation of compression waves is possible in all directions, and for a particular direction, two waves can propagate with different direction of polarization.

These results being obtained in the inclined coordinate system, they are transposed in the orthogonal coordinate system and in function of the real direction of propagation ϕ, what gives, for the velocities and the associated angles of polarization :

$$c_{\pm}(\phi) = \sqrt{\frac{2\sin(\varphi)E_x r}{M_s}\frac{\sin^2(\varphi)\sqrt{1+\cot(\varphi)\cot(\phi)}}{\sqrt{1-r\tan^2(\alpha)\pm\sqrt{\Delta}}}}$$

$$\Phi_{\pm}(\phi) = \tan^{-1}\left(\frac{1+r\tan^2(\alpha)\cos(2\varphi)\pm\sqrt{\Delta}}{r\tan^2(\alpha)\sin(2\varphi)}\right)$$

$$r = E_y/E_x$$
$$\Delta = (1-r\tan^2(\alpha))^2 + 4r\tan^2(\alpha)\cos^2(\varphi)$$
$$\tan(\alpha) = \sin(\varphi)\tan(\phi) + \cos(\varphi)$$

On the figure 5, the velocities and the associated polarization angles are presented for two angles of inclination of the lattice ($\varphi = \pi/2$ and $\pi/4$). As directions of polarization and of propagation are different, these waves are shear-compression waves, frequently encountered in anisotropic media. The propagation of these waves is moreover non dispersive, but dramatically anisotropic.

Now, the case with local dynamics is presented on the orthogonal frame ($\varphi = \pi/2$). In the previous equation, the masses vary in function of the frequency:

$$E_x\,\partial_{xx}\tilde{U}^0 + (M_f + M_w f(\omega))\,\omega^2\,\tilde{U}^0 = 0 \qquad (x\ 0)'$$

$$E_y\,\partial_{yy}\tilde{V}^0 + (M_w + M_f f(\omega))\,\omega^2\,\tilde{V}^0 = 0 \qquad (y\ 0)'$$

The nature of the waves are similar to the static case (two waves with different polarization in all directions). What differs is the variation of the apparent mass $m(\omega)$ (Fig. 6) appearing in the equations. The consequences are: (1) the media becomes dispersive, (2) bandgaps appear around odd modes of bending of elements. This effects are full described in [8].

5. Study of honeycomb

The honeycomb (Fig. 2) is now studied. Here the particularity of the local geometry is the presence of two families of nodes. On the figure 2, squares symbolize *internal nodes* and circles the *main nodes*. The procedure is similar to the previous case, so the calculus are not detailed in the following. Only the main points are described, then the results are given.

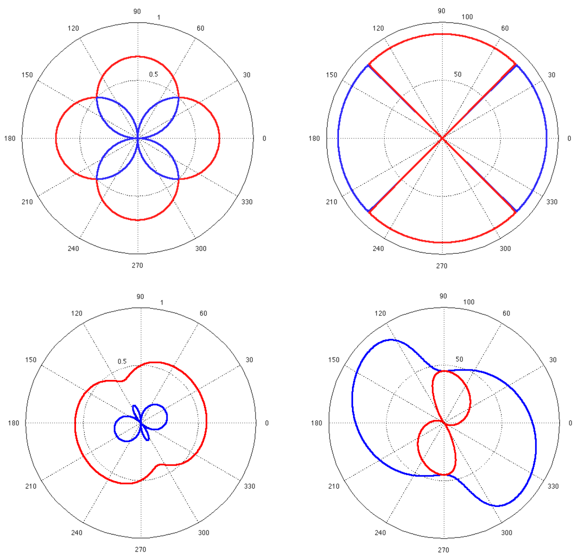

Figure 5. On top, case of the orthogonal inclined lattice ($\varphi = \pi/2$) and below, lattice with an inclination of $\varphi = \pi/4$. On the left, the dimensionless velocities in function of the angle of propagation ϕ (in degree) and on the right, the associated (blue or red) angle of polarization)

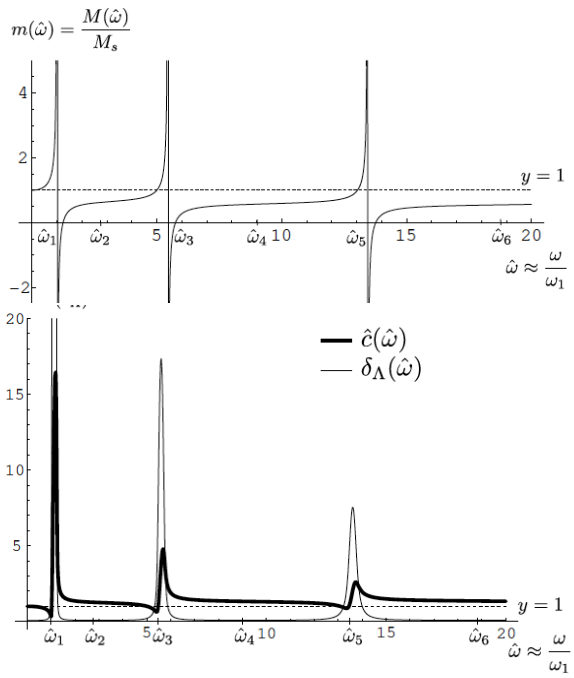

Figure 6. up: variation of the apparent mass induced by local resonance - down : variation of the velocity c and the associated attenuation δ in case of low viscous damping.

5.1. *Continuous description*

The reference coordinate system is here orthogonal. The first step consists in the resolution of the balance of internal nodes in function of the motions of the three main nodes of a cell. Then, the balance of a main node can be written in function of the six main nodes of the neighboring cells. As a consequence, the continuous description is given with, as variables, the motion (U, V and θ) of the main nodes. The motions of the internal node can then be deduced from them.

The rotation θ^0 of the main nodes are given by : $\theta^0(\vec{x}) = (\partial_x V^0(\vec{x}) - \partial_y U^0(\vec{x}))/2$. This expression is introduced in the equations and the homogenized description can be obtained, for the three first order, with $\vec{U}^i = (U^i, V^i)$:

$$\begin{cases} \lambda\vec{\nabla}(\nabla\vec{U}^0) = 0 \\ \lambda\vec{\nabla}(\nabla\vec{U}^1) = \vec{S}^1(\vec{U}^0) \\ \lambda\vec{\nabla}(\nabla\vec{U}^2) + \mu \triangle \vec{U}^0 = \vec{S}^1(\vec{U}^1) + \vec{S}^2(\vec{U}^0) \end{cases}$$

with, as macroscopic elastic parameters: $\lambda = \frac{E}{2\sqrt{3}}\frac{a}{\ell}$ and $\mu = \frac{E}{\sqrt{3}}\left(\frac{a}{\ell}\right)^3$. The vectors $\vec{S}^1(\vec{U})$ and $\vec{S}^2(\vec{U})$ are source terms. This description is very similar to a 2D isotropic Cauchy medium (right parts of the equation), but the presence of the source terms generates effects making this behavior more exotic.

5.2. *Compression and shear waves*

The compression waves are obtained for circular frequency ω_0 of order $O(\omega_r)$, and by introducing the wave expression (5), the first order becomes:

$$M_s\omega_0^2\vec{U}^0 = \lambda k^2(\alpha)(\vec{n}_\alpha.\vec{U}^0)\vec{n}_\alpha \qquad (8)$$

The dispersion relation is given by $M_s\omega_0^2 = \lambda k^2(\alpha)$ and the directions of polarization and of propagation are identical. The properties of the compression wave are the same, at the first order, as the one of a Cauchy medium, and the velocity is : $c = \sqrt{\lambda/M_s}$.

The shear waves are obtained for circular frequency ω_0 of order $O(\epsilon\omega_r)$, and the first order can be reduced to $\vec{n}_\alpha.\vec{U}^0 = 0$ what means, as for classical shear waves, that direction of polarization is orthogonal to the direction of propagation. However, the higher orders lead to the non classical relation of dispersion (in velocity):

$$M_s c^4(\alpha) - \mu c^2(\alpha) = \frac{\lambda\ell^2}{16}\omega^2(\cos(6\alpha) - 1) \qquad (9)$$

For the shear wave, the medium is both dispersive and anisotropic. For illustrating this effect, the velocities are presented on the figure 7 in function of the direction of propagation and of the frequency. At low frequency, the propagation is quasi-isotropic, whereas at higher frequency, bulbous forms appear in six directions, meaning in this direction, the velocities increase. Conversely to the lattice, for which the shear waves were just generated by the bending of the local elements, here, the traction-compression of the beams have a contribution. This is this contribution that introduces the properties of dispersivity and anisotropy.

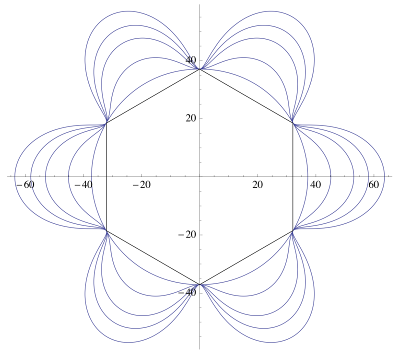

Figure 7. Profil of shear wave velocities in function of the direction of propagation and in function of the frequency. Results obtained on an aluminium honeycomb with the geometrical properties : ℓ =5.8 mm, h = 23 mm and a=0.06 mm. The bars represent the orientation of the honeycomb.

6. Conclusion

This work shows the interest of homogenization method of periodic discrete media (HPDM) for the study of the properties of discrete structures. Its main advantage is the analytical formulation which enables to understand the mechanisms governing the global behavior. This can also be used for the design of such materials. This work shows also the interest of this type of micro-structured materials, that present large variety of behavior, useful for the construction of directional or/and frequency filters.

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