# Numerical integration methods for the solution of Helmholtz equations with the Wave Based Technique 

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## Summary

The Wave Based Technique [7] is a method for airborne noise simulation, which is aimed for applications in the low- and mid-frequency range. The focus of this article is on application to sound radiation of internal combustion engines. The discretization of the Helmholtz equation, which is the basis for the Wave Based Technique, requires the computation of certain surface integrals with high accuracy using numerical quadrature. Structural vibrations of a given surface (e.g. the surface of an engine block), have to be mapped to the integration points. With increasing frequency the number of integration points in these quadratures has to be increased to achieve the desired accuracy. Every time the integration points are increased, the mapping procedure has to be done anew. To save computation time it is desirable to reuse the integration points and the already computed mapping information whenever this is possible. This paper demonstrates a strategy to calculate part of the mapping information in advance in order to increase the performance. The application of these strategies is benchmarked using two engine block models of different size. It is shown that the proposed strategy can lead to a significant reduction in computation time for models in which the mapping effort with the classical strategy is large, while maintaining the accuracy.


Figure 1. Helmholtz radiation problem

## 1. The Wave Based Technique for unbounded acoustic problems

### 1.1. Mathematical formulation

This article considers a 3-dimensional exterior acoustic radiation problem (Figure 1) on a domain $\Omega=$ $\mathbb{R}^{3}-{\overline{\Omega_{+}}}^{1}$, where $\Omega_{+} \subset \mathbb{R}^{3}$ is a bounded domain, refered to as scattering obstacle.
In absence of additional sound sources the acoustic sound pressure $p(\mathbf{r}, t)$ satisfies the wave equation

$$
\begin{equation*}
\Delta p-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} p=0 \tag{1}
\end{equation*}
$$

where $c$ denotes the speed of sound, cf. [4]. Since a solution of (1) should be determined by its Fourier in-

[^0]tegral $p(\mathbf{r}, t)=\int_{-\infty}^{\infty} p_{\omega}(\mathbf{r}) e^{i \omega t} d \omega$, it suffices to study time-harmonic solutions $p(\mathbf{r}, t)=p_{\omega}(\mathbf{r}) e^{i \omega t}$, with circular frequency $\omega$. Substituting into (1) yields that the steady-state pressure $p_{\omega}(\mathbf{r})$ satisfy
\[

$$
\begin{equation*}
\Delta p_{\omega}+k^{2} p_{\omega}=0 \tag{2}
\end{equation*}
$$

\]

This is Helmholtz' equation with wave number $k=\frac{\omega}{c}$. The particle velocity $\mathbf{v}(\mathbf{r}, t)$ has the same timeharmonicity $\mathbf{v}(\mathbf{r}, t)=\mathbf{v}_{\omega}(\mathbf{r}) e^{i \omega t}$ as the sound pressure. The steady-state particle velocity $\mathbf{v}_{\omega}$ satisfies

$$
\begin{equation*}
\mathbf{v}_{\omega}(\mathbf{r})=\frac{i}{\omega \rho} \nabla p_{\omega}(\mathbf{r}) \tag{3}
\end{equation*}
$$

where $\rho$ is the ambient fluid density.
Different boundary condition types will be considered. It is assumed that the boundary of $\Omega$ is partitioned as $\Gamma:=\partial \Omega=\Gamma_{p} \uplus \Gamma_{v} \uplus \Gamma_{Z}$ as depicted in Figure 1, on which the following boundary conditions are imposed.

- On $\Gamma_{p}$ a pressure boundary condition

$$
\begin{equation*}
p_{\omega}(\mathbf{r})=p_{b}(\mathbf{r}) \quad \text { for each } \mathbf{r} \in \Gamma_{p} \tag{4}
\end{equation*}
$$

for a given boundary pressure function $p_{b}$ on $\Gamma_{p}$.

- On $\Gamma_{v}$ a normal velocity boundary condition

$$
\begin{equation*}
\mathbf{n}^{T} \mathbf{v}_{\omega}(\mathbf{r})=v_{b}(\mathbf{r}) \quad \text { for each } \mathbf{r} \in \Gamma_{v} \tag{5}
\end{equation*}
$$

for a given boundary normal velocity function $v_{b}$ on $\Gamma_{v}$ and the outward normal vector ${ }^{2} \mathbf{n}$ on $\partial \Omega$.

- On $\Gamma_{Z}$ a normal impedance boundary condition

$$
\begin{equation*}
\mathbf{n}^{T} \mathbf{v}_{\omega}(\mathbf{r})-\frac{1}{Z_{b}(\mathbf{r})} p_{\omega}(\mathbf{r})=0 \quad \text { for each } r \in \Gamma_{Z} \tag{6}
\end{equation*}
$$

for a given normal impedance function $Z_{b}$ on $\Gamma_{Z}$.

[^1]- In addition the Sommerfeld radiation condition

$$
\begin{equation*}
\frac{\partial}{\partial r} p_{\omega}(\mathbf{r})+i k p_{\omega}(\mathbf{r})=o\left(r^{-1}\right) \quad \text { as } r \rightarrow \infty \tag{7}
\end{equation*}
$$

is imposed uniformly in all directions ${ }^{3}$. Physically it means that the acoustic wave is not reflected at infinity. It ensures that the boundary value problem is well-posed on unbounded domains. For details the reader is refered to $[3,4]$.

### 1.2. Solution using the wave based technique

The idea of the Wave Based Technique (WBT) is to decompose the domain $\Omega$ into subdomains, to impose additional smoothness boundary conditions on the subdomain interfaces, and to use a linear combination of exact solutions of the Helmholtz equation (2) on each subdomain as an ansatz. Subsequently the boundary conditions are enforced by an appropriate choice of parameters in this ansatz. For further information on WBT the reader is refered to [7, 2]
The first step is to decompose $\Omega$ into a bounded and an unbounded part by introducing an artificial truncation boundary $\Gamma_{T}$. The bounded and unbounded part are denoted by $\Omega_{B}$ and $\Omega_{U}$ respectively. The bounded part $\Omega_{B}$ is decomposed into subdomains $\Omega_{1}, \ldots, \Omega_{N}$. To ensure convergence of the wave based method the geometry of these subdomains has to be restricted ${ }^{4}$. For the pressure inside the domain $\Omega_{\alpha}$ the ansatz

$$
\begin{equation*}
p_{\alpha}(\mathbf{r})=\sum_{i=1}^{n_{\alpha}} \sum_{j=1}^{3} p_{i j}^{(\alpha)} \Phi_{i j}^{(\alpha)}(\mathbf{r}) \tag{8}
\end{equation*}
$$

is used. The parameter $n_{\alpha}$ depends on $\Omega_{\alpha}$ and on the frequency $\omega$. The functions $\Phi_{i j}^{(\alpha)}$, given by

$$
\Phi_{i j}^{(\alpha)}(\mathbf{x}):= \begin{cases}e^{-i k_{i x}^{(\alpha)} x} \cos \left(k_{i y}^{(\alpha)} y\right) \cos \left(k_{i z}^{(\alpha)} z\right) & j=1  \tag{9}\\ \cos \left(k_{i x}^{(\alpha)} x\right) e^{-i k_{i y}^{(\alpha)} y} \cos \left(k_{i z}^{(\alpha)} z\right) & j=2 \\ \cos \left(k_{i x}^{(\alpha)} x\right) \cos \left(k_{i y}^{(\alpha)} y\right) e^{-i k_{i z}^{(\alpha)} z} & j=3\end{cases}
$$

are called wave functions. In order for $\Phi_{i j}^{(\alpha)}$ to be an exact solution of the Helmholtz equation (2) it is necessary and sufficient that the equation

$$
\begin{equation*}
\left(k_{i x}^{(\alpha)}\right)^{2}+\left(k_{i y}^{(\alpha)}\right)^{2}+\left(k_{i z}^{(\alpha)}\right)^{2}=k^{2} \tag{10}
\end{equation*}
$$

be satisfied.
The pressure expansion in the domain $\Omega_{U}$ reads

$$
\begin{equation*}
p_{U}(\mathbf{r})=\sum_{l=0}^{n_{U}} \sum_{m=-l}^{l} p_{l m}^{(U)} \Phi_{l m}^{(U)}(\mathbf{r}) \tag{11}
\end{equation*}
$$

[^2]The functions

$$
\begin{equation*}
\Phi_{l m}^{(U)}(r, \phi, \theta):=h_{l}(k r) Y_{l}^{m}(\phi, \theta) \tag{12}
\end{equation*}
$$

are called radiation functions ${ }^{5}$. They satisfy the Helmholtz equation on $\Omega_{U}$ and also the Sommerfeld radiation condition (7), cf. [4]. The parameter $n_{U}$ again depends on the frequency.
For $\alpha=1, \ldots, N, U^{6}$ the notations $\Gamma_{\alpha p}:=\Gamma_{p} \cap \Omega_{\alpha}$, $\Gamma_{\alpha v}:=\Gamma_{v} \cap \Omega_{\alpha}$ and $\Gamma_{\alpha Z}:=\Gamma_{Z} \cap \Omega_{\alpha}$ are used. Furthermore the following residuals are defined to enforce the boundary conditions (4), (5) and (6).

$$
\begin{gather*}
R_{p}^{(\alpha)}(\mathbf{r}):=p_{\alpha}(\mathbf{r})-p_{b}(\mathbf{r}) \quad \text { for } \mathbf{r} \in \Gamma_{\alpha p}  \tag{13}\\
R_{v}^{(\alpha)}(\mathbf{r}):=\frac{i}{\rho \omega} \mathbf{n}^{T} \nabla p_{\alpha}(\mathbf{r})-v_{b}(\mathbf{r}) \quad \text { for } \mathbf{r} \in \Gamma_{\alpha v}  \tag{14}\\
R_{Z}^{(\alpha)}(\mathbf{r}):=\frac{i}{\rho \omega} \mathbf{n}^{T} \nabla p_{\alpha}(\mathbf{r})-\frac{1}{Z_{b}(\mathbf{r})} p_{\alpha}(\mathbf{r}) \quad \text { for } \mathbf{r} \in \Gamma_{\alpha Z} \tag{15}
\end{gather*}
$$

At the domain interfaces $\Gamma_{\alpha \beta}:=\overline{\Omega_{\alpha}} \cap \overline{\Omega_{\beta}}$ the residuals

$$
\begin{align*}
R^{(\alpha, \beta)}(\mathbf{r}):= & \frac{i}{\rho \omega} \mathbf{n}^{T} \nabla\left(p_{\alpha}(\mathbf{r})+p_{\beta}(\mathbf{r})\right)  \tag{16}\\
& +\frac{1}{Z_{c}}\left(p_{\alpha}(\mathbf{r})-p_{\beta}(\mathbf{r})\right) \quad \text { for } \mathbf{r} \in \Gamma_{\alpha \beta}
\end{align*}
$$

are introduced. The parameter $Z_{c}$ weights the relative importance of the pressure term against the normal velocity term.
For brevity's sake the notation

$$
\begin{equation*}
\langle f, g\rangle_{\Gamma}:=\int_{\Gamma} f g d S \tag{17}
\end{equation*}
$$

is employed for surface integrals.
Instead of requiring that the residuals vanish, it is merely demanded that they vanish in average. This is expressed in the weighted residual formulation, which reads

$$
\begin{align*}
\left\langle\mathbf{n}^{T} \nabla q, R_{p}^{(\alpha)}\right\rangle_{\Gamma_{\alpha p}} & +\left\langle q, R_{v}^{(\alpha)}\right\rangle_{\Gamma_{\alpha v}}+\left\langle q, R_{Z}^{(\alpha)}\right\rangle_{\Gamma_{\alpha Z}} \\
& +\sum_{\beta \neq \alpha}\left\langle q, R^{(\alpha, \beta)}\right\rangle_{\Gamma_{\alpha \beta}}=0 \tag{18}
\end{align*}
$$

for $\alpha=1, \ldots, N, U$ and all $q$ from an appropriate class of test functions ${ }^{7}$.

[^3]By plugging the expansions (8) and (11) into equation (18) and using $\Phi_{i j}^{(\alpha)}$ with $\alpha=1, \ldots, N, U$ as test functions, the following linear system is obtained.

$$
\begin{equation*}
\sum_{k=1}^{n_{\alpha}} \sum_{l} a_{i j k l}^{(\alpha)} p_{k l}^{(\alpha)}+\sum_{\beta \neq \alpha} \sum_{k=1}^{n_{\alpha}} \sum_{l} c_{i j k l}^{(\alpha, \beta)} p_{k l}^{(\beta)}=b_{i j}^{(\alpha)} \tag{19}
\end{equation*}
$$

where the sums over $l$ run through $1,2,3$ for $\alpha=$ $1, \ldots, N$ and through $-k, \ldots, k$ for $\alpha=U$ and the coefficients $a_{i j k l}^{(\alpha)}, c_{i j k l}^{(\alpha, \beta)}$ and $b_{i j}^{(\alpha)}$ are given by

$$
\begin{align*}
a_{i j k l}^{(\alpha)}= & \left\langle\mathbf{n}^{T} \nabla \Phi_{i j}^{(\alpha)}, \Phi_{k l}^{(\alpha)}\right\rangle_{\Gamma_{\alpha p}}+\frac{i}{\rho \omega}\left\langle\Phi_{i j}^{(\alpha)}, \mathbf{n}^{T} \nabla \Phi_{k l}^{(\alpha)}\right\rangle_{\Gamma_{\alpha v}} \\
& +\frac{i}{\rho \omega}\left\langle\Phi_{i j}^{(\alpha)}, \mathbf{n}^{T} \nabla \Phi_{k l}^{(\alpha)}\right\rangle_{\Gamma_{\alpha Z}}-\left\langle\Phi_{i j}^{(\alpha)}, \frac{\Phi_{k l}^{(\alpha)}}{Z_{b}}\right\rangle_{\Gamma_{\alpha Z}} \\
& +\frac{i}{\rho \omega} \sum_{\beta \neq \alpha}\left\langle\Phi_{i j}^{(\alpha)}, \mathbf{n}^{T} \nabla \Phi_{k l}^{(\alpha)}\right\rangle_{\Gamma_{\alpha \beta}} \\
& +\frac{1}{Z_{c}} \sum_{\beta \neq \alpha}\left\langle\Phi_{i j}^{(\alpha)}, \Phi_{k l}^{(\alpha)}\right\rangle_{\Gamma_{\alpha \beta}} \tag{20}
\end{align*}
$$

$$
\begin{equation*}
c_{i j k l}^{(\alpha, \beta)}=\left\langle\Phi_{i j}^{(\alpha)}, \mathbf{n}^{T} \nabla \Phi_{k l}^{(\beta)}\right\rangle_{\Gamma_{\alpha \beta}}-\frac{1}{Z_{c}}\left\langle\Phi_{i j}^{(\alpha)}, \Phi_{k l}^{(\beta)}\right\rangle_{\Gamma_{\alpha \beta}} \tag{21}
\end{equation*}
$$

$$
\begin{equation*}
b_{i j}^{(\alpha)}=\left\langle\mathbf{n}^{T} \nabla \Phi_{i j}^{(\alpha)}, p_{b}\right\rangle_{\Gamma_{\alpha p}}+\left\langle\Phi_{i j}^{(\alpha)}, v_{b}\right\rangle_{\Gamma_{\alpha v}} \tag{22}
\end{equation*}
$$

By defining

$$
\begin{align*}
& A_{\alpha}:={\underset{i j k l}{(\alpha)}{\underset{(i, j)}{(k, l)}}^{(\alpha)}}^{C_{\alpha \beta}:=\left(c_{i j k l}^{(\alpha, \beta)}\right)_{(i, j)}^{(k, l)}}  \tag{23}\\
& \mathbf{b}_{\alpha}:=\left(b_{i j}^{(\alpha)}\right)_{(i, j)} \mathbf{p}_{\alpha}:=\left(p_{i j}^{(\alpha)}\right)_{(i, j)}
\end{align*}
$$

the linear system (19) can be written in matrix form:

## 2. Calculation of the WBT matrix

### 2.1. Numerical Integration

In order to compute the coefficients (20), (21) and (22) integrals over a surface $\Gamma \subset \mathbb{R}^{3}$ have to be evaluated. For details on numerical integration the reader is refered to [8]. $\Gamma$ is assumed to be given in terms of a parametrization $\mathbf{x}:[-1,1]^{2} \longrightarrow \Gamma \subset \mathbb{R}^{3}$.
An integration rule for the surface $\Gamma$ can be obtained via pullback of a 2 -dimensional integration rule

$$
\begin{equation*}
\int_{[-1,1]^{2}} F d \xi d \eta \approx \sum_{i=0}^{n} \omega_{i} F\left(\xi_{i}, \eta_{i}\right) \tag{25}
\end{equation*}
$$

for the reference quadrangle $[-1,1]^{2}$, along the parametrization $\mathbf{x}$ :

$$
\begin{align*}
\int_{\Gamma} F d S & =\int_{[-1,1]^{2}}(F \circ \mathbf{x})\left\|\frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}\right\| d \xi d \eta \\
& \approx \sum_{i=0}^{n} \omega_{i}\left\|\frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}\right\|\left(\xi_{i}, \eta_{i}\right) F\left(\mathbf{x}\left(\xi_{i}, \eta_{i}\right)\right) \\
& =\sum_{i=0}^{n} \omega_{i}^{\prime} F\left(x_{i}, y_{i}, z_{i}\right) \tag{26}
\end{align*}
$$

with new integration points and weights given by $\left(x_{i}, y_{i}, z_{i}\right):=\mathbf{x}\left(\xi_{i}, \eta_{i}\right)$ and $\omega_{i}^{\prime}:=\omega_{i}\left\|\frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}\right\|\left(\xi_{i}, \eta_{i}\right)$.
The above mentioned 2-dimensional integration scheme for the reference quadrangle can be constructed out of a 1-dimensional integration rule

$$
\begin{equation*}
\int_{[-1,1]} F(\xi) d \xi \approx \sum_{i=0}^{m} \alpha_{i} F\left(\xi_{i}\right) \tag{27}
\end{equation*}
$$

by the tensor product construction

$$
\begin{equation*}
\int_{[-1,1]^{2}} F d \xi d \eta \approx \sum_{i=0}^{m} \sum_{j=0}^{m} \alpha_{i} \alpha_{j} F\left(\xi_{i}, \xi_{j}\right) . \tag{28}
\end{equation*}
$$

For the 1-dimensional rule (27) the following will be considered:

Gauss-Legendre rule For the ( $n+1$ )-point GaussLegendre quadrature the integration points $\xi_{i}$ are the zeros of the Legendre polynomial $P_{n}$. The weights are

$$
\begin{equation*}
\alpha_{i}=\frac{2}{\left(1-x_{i}^{2}\right) P_{n}^{\prime}\left(x_{i}\right)^{2}} . \tag{29}
\end{equation*}
$$

The Gauss-Legendre rule is exact for polynomials of degree $2 n+1$ or less.

Simpson's rule For even $n$ the integration nodes are $\xi_{i}=-1+\frac{2}{n} i$ for $i=0, \ldots, n$. The weights are

$$
\alpha_{i}= \begin{cases}\frac{8}{3 n} & i=1,3, \ldots, n-1  \tag{30}\\ \frac{4}{3 n} & i=2,4, \ldots, n-2 \\ \frac{2}{3 n} & i=0, n\end{cases}
$$

Simpson's rule is exact for polynomials of degree $\leq 3$.

### 2.2. Inverse Distance Weight Mapping

In practice the problem geometry is frequently given as a Finite Elements model and the boundary values are given at the FE nodes. In order to calculate
the integrals for WBT, knowledge of the boundary values in the integration points is needed. These values can be approximated by an interpolation method called Inverse Distance Weighting (IDW) [9]. A brief describtion of this method is given here.
A sufficently "well-behaved" function $f: D \longrightarrow \mathbb{R}$ whose values $f_{i}:=f\left(x_{i}\right)$ are known only in certain sample points $x_{0}, \ldots, x_{n} \in D$, can be approximated in a query point $x \in D$ by
$\bar{f}(x):= \begin{cases}\frac{1}{\sum_{i=0}^{n} w_{i}(x)} \sum_{i=0}^{n} w_{i}(x) f_{i} & x \notin\left\{x_{0}, \ldots, x_{n}\right\} \\ f_{i} & x=x_{i}\end{cases}$
with the weights $w_{i}(x)=\frac{1}{\left\|x-x_{i}\right\|^{p}}$. Here $p$ is a positive constant, which controls how much influence the close sample points have on the approximation. If $p$ is increased, the weights of sample points farther away will be small, thus decreasing their contribution to (31).
Normally the number of points $n$ is very large, so it might be inefficient to calculate (31). Instead a search for the $k$ points among $x_{0}, \ldots, x_{n}$ closest to $x$ is performed first and then (31) is modified to only sum up those points. This step can be done using a data structure which supports efficient $k$-nearest neighbor searches, like $k d$-trees [1] or Octrees [6].

## 3. Comparison of different methods

### 3.1. Workflow

In this subsection a short describtion of the applied worflow is given. The model geometry is given as Finite Element mesh with prescribed boundary values in the nodes. The FE model is enclosed by a rectangular bounding box, which in turn lies inside the truncation sphere. The part of the engine exterior, which lies inside the bounding box, is partitioned into rectangular domains. The part between the bounding box and the truncation sphere is partitioned into six sphere caps, which are called spherical adaptation elements.
A full simulation consists of the calculation of a discrete range of frequency steps. For each frequency several steps are executed:

1. Generation and setup of the integration points and weights, as described in subsection 2.1.
2. IDW Mapping of the boundary values from the FE nodes to the integration points, see subsection 2.2.
3. Calculation of the entries of the WBT matrix (20), (21) and the right-hand-side (22).
4. Solution of the WBT system (24).
5. Post-Processing, e.g. computation of the pressure and velocity fields at predefined response points.

### 3.2. Description of the test cases

Case 1 The Gauss-Legendre rule together with an incremental increse of the number of integration
points for each surface per frequency is used. In some frequency steps the integration points might not change. In this case the $k$-nearest neighbor data from the mapping is preserved from the previous frequency and used to calculate (31) for the current frequency.

Case 2 Like Case 1, but instead of Gauss-Legendre, Simpson's rule is used. In each frequency step the same number of integration points as in Case 1 is used.

Case 3 Here the Gauss-Legendre rule is used, but instead of increasing the number of integration points with each frequency, the same number of integration points is used for all frequencies, namely the one used for the highest frequency in the other two cases. Altough this will increase the times needed for the computation of the coefficients (20), (21) and (22) in all frequencies except the highest, this has two important advantages. Since the integration points do not change, recalculation of the $k$-nearest neighbor information in the IDW mapping is never required and it also eases the post-processing of the results since for all frequencies the same nodes are used. However it is a priori not clear that the choice of the highest frequency in the range does not influence the results. For example, does the computation yield the same result at 1000 Hz , if we compute it once using 2000 Hz as upper bound and once using 3000 Hz as upper bound. A similar question arises, if one calculates up to a certain frequency and later decides to continue the calculation for frequencies above. In this case the mapping has to be recalculated before continuing.

### 3.3. Description of the used models

The proposed strategies are validated using two industrial engine models. The model geometries are given as Finite Element meshes with prescribed velocity excitations in the nodes. The simulations for both models were performed from a certain start frequency with a constant frequency step size up to an end frequency.
Table I summarises the details on the models.
Model 1 is an inline 4-cylinder Diesel engine with relatively small number of FE nodes. It is to be expected that the IDW mapping takes only a small part of the overall computation time.

Model 2 is an inline 4-cylinder gasoline engine with high number of FE nodes. IDW mapping for this model is expected to be rather time consuming.

### 3.4. Results

The calculations are OpenMP parallelized and performed on a $\operatorname{Intel}(\mathrm{R})$ Core(TM) i7-4900MQ 2.80 GHz CPU with 16.0 GB RAM. Six cores are used.
In Case 3 the highest simulation frequency has a direct influence on the number of integration points used on

Table I．Key information about the used models

|  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| M1 | 33643 | 30 | 171 | 25 | 3000 | 25 |
| $\overline{\mathrm{M} 2}$ | 442051 | 43 | 273 | 500 | 3000 | 50 |



Figure 2．Sound Pressure Level comparison for different upper frequency bounds（Model 2）
each surface．Hence it is first confirmed that using a different upper frequency bound in the calculation leaves the results invariant．To confirm this，strategy 3 is performed with different upper frequency bounds： $1000 \mathrm{~Hz}, 2000 \mathrm{~Hz}$ ，and 3000 Hz ．Figure 2 shows the Sound Pressure Level at a response point， 1 m above the engine of Model 2．The results are compared in the overlaps of these ranges and show that the choice of the frequency bound barely influences the result．
Figure 3 shows the pressure response at a response point in a distance of 1 m above the engine．In direct comparison between cases 1,2 and 3 ，all three strate－ gies yield very similar results for the exterior pressure field except for small deviations，but the relative error lies well below $2 \%$ ．
The performance results are summarised in Figures 4， 5 and table II．The terms IDW Mapping，Matrix and Solve refer to steps 2 ．to 4 ．of subsection 3．1．The run－time differences between strategies 1 and 2 are negligible in both models．As expected the IDW map－ ping in Cases 1 and 2 is rather inexpensive for Model 1．Here the use of strategy 3 does even increase the computational time needed about $7 \%$ ．In Model 2 the IDW mapping consumes about two fifths of the over－ all computation time in Cases 1 and 2，but less than one sixth in Case 3，thus reducing the total simulation time by about $15 \%$ from 1 h 38 m to 1 h 19 m ．
The tooth shape of the graphs in Cases 1 and 2 of Figure 5 results from the fact that in some frequency steps it is not necessary to increase the number of in－ tegration points to achieve the desired accuracy．Thus the IDW mapping is reused at these frequencies．This can also be observed in Model 1，but the IDW map－

Table II．Performance results（in s）

|  | 合 | $\begin{aligned} & \text { 希 } \\ & \text { 节 } \end{aligned}$ | O | F |
| :---: | :---: | :---: | :---: | :---: |
| Model 1 Case 1 | 91 | 2664 | 1579 | 4334 |
| Model 1 Case 2 | 93 | 2715 | 1653 | 4461 |
| Model 1 Case 3 | 36 | 3121 | 1526 | 4683 |
| Model 2 Case 1 | 2361 | 2274 | 1282 | 5917 |
| Model 2 Case 2 | 2517 | 2216 | 1265 | 5998 |
| Model 2 Case 3 | 765 | 2756 | 1222 | 4743 |

ping in this model takes so few time that one can barely see the tooth shape in Figure 4.

## 4．Conclusion

It has been shown that calculating the mapping infor－ mation in advance for a large number of integration points can increase the performance of WBT calcula－ tions for models with large number of FE nodes，while it might be disadvantageous in small models．In any case the accuracy of the results is maintained．

## Acknowledgement

This work is funded by the European Commission ITN Marie Curie proj．No． 605867 ＂BATWOMAN＂ （Basic Acoustics Training－\＆Workprogram on Methodologies for Acoustics－Network）．

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Figure 3. Sound Pressure Level


Figure 4. Performance results Model 1: Total run-time (above) and run-time per frequency (below)


Figure 5. Performance results Model 2: Total run-time (above) and run-time per frequency (below)


[^0]:    ${ }^{1}$ The notation $\bar{\Omega}$ means the topological closure of $\Omega$, i.e. $\Omega$ together with its boundary.

[^1]:    ${ }^{2}$ Outward-pointing is meant as "out of $\Omega$ " here, but into the obstacle $\Omega_{+}$.

[^2]:    ${ }^{3}$ Boldface letters are used to denote vectors. The plain letter $r$ denotes the length of the vector $\mathbf{r}$.
    ${ }^{4}$ Convex domains are admissible, although more general domains could be allowed, as well, cf. [5].

[^3]:    ${ }^{5}$ Here $h_{l}$ are spherical Hankel functions and $Y_{l}^{m}$ are spherical harmonics.
    ${ }^{6}$ To treat the domains $\Omega_{1}, \ldots, \Omega_{N}$ and $\Omega_{U}$ in a coherent way, the index $\alpha$ runs through the numbers $1, \ldots, N$ and $U$. Here $U$ is merely a symbol, not a number.
    ${ }^{7}$ It will be assumed this class contains at least the wave functions (9) and the radiation functions (12).

