Development of the partition of unity finite element method for the 3D analysis of interior sound fields

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Summary
This work is concerned with the numerical simulation of sound pressure field in three-dimensional cavities in which absorbing materials are present. Standard techniques such as the Finite Element Method are known to be extremely demanding computationally when the frequency increases and thus limited to low frequency applications. To alleviate these difficulties, an alternative formulation based on the Partition of Unity Finite Element Method is proposed. The method involves enriching the approximation finite element space by expanding the acoustic pressure in a set of plane waves propagating in various directions over the unit sphere. Particular attention is devoted to the fast and accurate computation of highly oscillating integrals which is required by the method. Convergence studies show that these wave finite elements allow to capture accurately the wave field with a number of degrees of freedom that only grows quadratically with the frequency yielding drastic data reduction compared to classical FEM. Results of practical interest are shown for the case of a sound source placed in a reverberation room with absorbing materials.

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1. Introduction

Traditional Finite Element Method (FEM) is not efficient enough to solve medium and high frequency acoustic waves because of excessive demands involving heavy computational cost. This limitation results from the thumb rule which indicates that at least 10 nodal points per wavelength are required with classical FEM. Therefore, new deterministic prediction techniques have been developed in the recent years to overcome this limitation, most of them are implemented by expanding the dynamic field variable with a set of oscillatory wave functions which are the analytical solutions to the governing equation of the problem. These techniques include the Partition of Unity Finite Element Method (PUFEM) [1], the Ultra-Weak formulation [2], Wave-Based Methods [3], the Discontinuous Galerkin Method [4] and the Variational Theory of Complex Rays [5]. All of these methods can offer a drastic reduction in degrees of freedom compared with conventional FEM. Among them, PUFEM offers the advantage of being very similar to the FEM, can be easily adapted to any FEM mesh and has been successfully used to solve acoustic wave scattering in 2 and 3 dimensions [6] [7], flow acoustic and other wave propagation problems [8] [9] [10].

Although these prediction techniques allows a huge reduction in the number of degrees of freedom, the computation of element matrices requires the integration of highly oscillatory functions leading to expensive computational costs due to the need of a too large number of integration points. These limitations can be prohibitive especially for 3D simulations. Therefore, several new integration schemes were developed to improve competitiveness of the PUFEM. Sugimoto [11], Bettess [12] and Gordon [13] used the divergence theorem to obtain the exact integration over a polygon. Gabard [14] extended this method to volume integrals. El Kacimi [15] presented a similar approach in the evaluation of element matrices for elastic wave scattering problems. However, to the authors' knowledge no results appear in the literature regarding appropriate integration techniques for the computation of PUFEM element matrices in 3-dimensional domains.

In this paper, a new exact integration scheme over a tetrahedron element is proposed and some numerical aspects of this scheme are investigated. The derivation of analytical integration of element matrices is explained. Through intensive convergence tests, it is shown that the PUFEM elements allow to capture accurately the wave field with a number of degrees of freedom that only grows quadratically with the frequency, yielding drastic data reduction compared to classical FEM. Results of practical interest are shown for the case of a sound source placed in a reverberation room with absorbing materials on the wall.

2. PUFEM for 3D acoustic waves

The wave governing equation for the acoustic pressure p is the classical Helmholtz equation,

$$\Delta p + k^2 p = 0 \tag{1}$$

where $k = \omega/c$ is the wavenumber and ω is the harmonic frequency In order to get the numerical solution to this problem, we need multiply equation (1) with a weighting function δp, so that the weak form of the Helmholtz equation can be written as,

$$\int_\Omega \left(\nabla p \cdot \nabla(\delta p) - k^2 p \delta p\right) \mathrm{d}\Omega - \int_\Gamma \frac{\partial p}{\partial n} \delta p \mathrm{d}\Gamma = 0 (2)$$

where Ω is a closed three-dimensional domain with boundary Γ. The PUFEM consists in expanding the pressure using a set of plane waves to enrich the classical finite element approximation space as follows:

$$p(\boldsymbol{x}) = \sum_{j=1}^{4} N_j \sum_{q=1}^{Q_j} A_{jq} \exp(\mathrm{i}k\boldsymbol{d}_{jq} \cdot \boldsymbol{x}) \tag{3}$$

where $\boldsymbol{d}_{jq}$ denote the directional vectors of the plane waves basis attached to node j. The distribution of plane waves directions is based on the equal spacing of points on the unit sphere [16] as shown in Figure 1. Coefficients A_{jq} represent the amplitude of each plane wave and are the unknowns of the problem.

After substitution of the plane wave expansion in the weak formulation, we find that PUFEM matrix coefficients are obtained from the sum of 4 integrals over an element Ω_e:

$$\begin{aligned} I = &-\kappa^2(1 + \boldsymbol{d}' \cdot \boldsymbol{d}'') \int_{\Omega_e} N_p N_q \phi \,\mathrm{d}\Omega \\ &+ \mathrm{i}\kappa \boldsymbol{d}' \cdot \nabla N_q \int_{\Omega_e} N_p \phi \,\mathrm{d}\Omega \\ &+ \mathrm{i}\kappa \boldsymbol{d}'' \cdot \nabla N_p \int_{\Omega_e} N_q \phi \,\mathrm{d}\Omega \\ &+ \nabla N_p \cdot \nabla N_q \int_{\Omega_e} \phi \,\mathrm{d}\Omega, \end{aligned} \tag{4}$$

where we put $\phi = \exp(\mathrm{i}\kappa \boldsymbol{d} \cdot \boldsymbol{x})$ with $\kappa = k|\boldsymbol{d}' + \boldsymbol{d}''|$ and $\boldsymbol{d} = (\boldsymbol{d}' + \boldsymbol{d}'')/|\boldsymbol{d}' + \boldsymbol{d}''|$. Here symbols $'$ and $''$ refers to trial and test functions. The next step is to take advantage of the fact that shape functions are linear over the tetrahedron finite element. This signifies that, application of the Green theorem on volume integrals gives the following result

$$\int_{\Omega_e} F\,\phi = -\int_{\partial\Omega_e} (\epsilon F \boldsymbol{d} + \epsilon^2 \nabla F + \epsilon^3 \triangle F \boldsymbol{d}) \cdot \boldsymbol{n}\,\phi$$

where $\epsilon = \mathrm{i}/\kappa$ and F stands for a constant, a linear or a quadratic function with respect to local coordinates $\boldsymbol{x} = \boldsymbol{x}(\zeta_1, \zeta_2, \zeta_3)$ (the linear mapping between real and local space is illustrated in Fig. 2). Following the same strategy, the surface integral can be further reduced to line integrals which can be reduced to a closed form expression. Finally, we arrive at analytical formulas requiring only the information about the 4 vertices of the tetrahedron element, thus avoiding the use of high-order numerical quadrature.

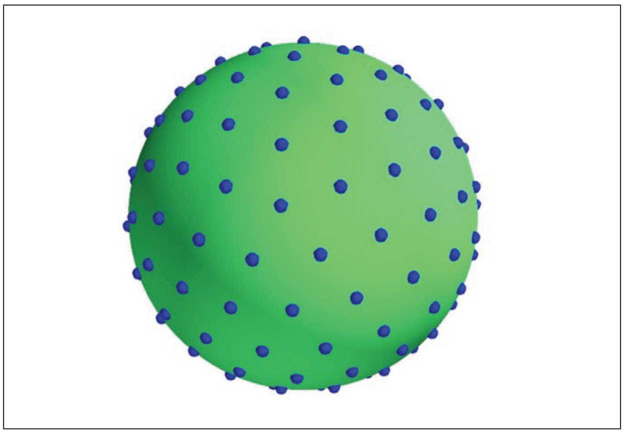

Figure 1. End points distribution of directional vectors of plane waves (taken from [16]).

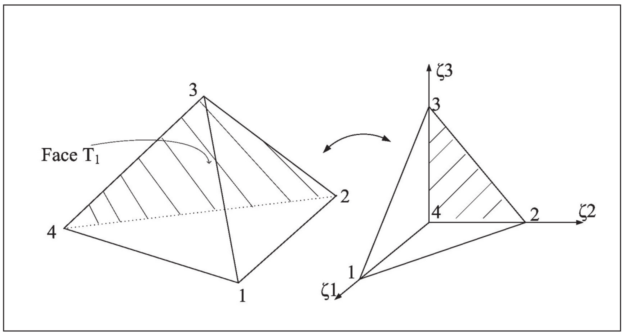

Figure 2. 3D coordinates systems in real and local element space.

3. Convergence test

The purpose of this section is to assess the numerical performances of the proposed technique in terms of accuracy and complexity. Here the idea is test the converge of the method towards an exact solution without modifying the coarse PUFEM mesh. In other words we perform a Q-refinement as opposed to a h-refinement. Through numerous numerical tests, it was observed the PUFEM accuracy depends mainly on 2 parameters: the element size, call it h_j defined as the longest edge attached to node j and the number of wavelengths spanned by the element. In the following, we may assume that the number of plane waves attached to each node should vary quadratically like $(kh_j)^2$ so we put

$$Q_j \approx C(kh_j)^2. \tag{5}$$

Coefficient C can also be viewed as a function of kh_j and it must be adjusted according to the configuration and expected accuracy. The behavior of C with respect to the nondimensional frequency kh_j can be found by choosing an artificial wave propagation problem for which an exact solution is easily available. To do this, we consider an arbitrary incident plane wave propagating inside a single isosceles tetrahedron (all edges of the tetrahedron are equal). To be fair in our convergence test, the incident plane wave direction is always chosen as far as possible from the plane waves directions of the PUFEM basis, in order to avoid peculiar behaviors. For the sake of illustration, Figure 3 illustrates the PUFEM solution for $kh = 50$ and $Q = 276$ plane wave basis are used per node (in total $4 \times 276 = 1104$ are used to simulate the arbitrary plane wave in the element). Figure 4 shows the relevant curves describing the convergence behaviors. The convergence rate for the tetrahedron was evaluated for two expected accuracy of 0.1% and 1%. Another similar scenario was tested with a regular cube comprising 24 elements. Here only the curve associated with 1% error accuracy is shown. For this specific case, it turns out that more plane waves are required (per node) in order to reach 1% accuracy compared with the tetrahedron case. This series of test show that coefficient C generally lies in the interval [0.1-0.7] as long as the frequency is sufficiently high compared to the element length and clearly the scenario $kh_j < 5$ should be avoided when using the PUFEM technology. On the other hand, high frequency calculations allows a substantial reduction in the number of degrees of freedom compared to classical FEM (for which it should grow at least cubically with frequency) as all curves show similar asymptotic behavior with $Q_j \sim 0.1(kh_j)^2$.

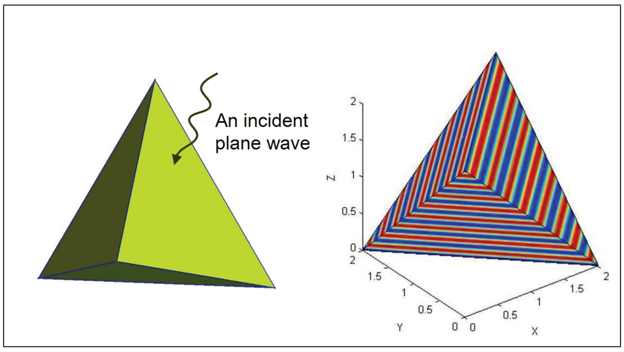

Figure 3. The model (left), PUFEM solution (right), $kh = 50$ and Q=276, CPU time < 3min Matlab.

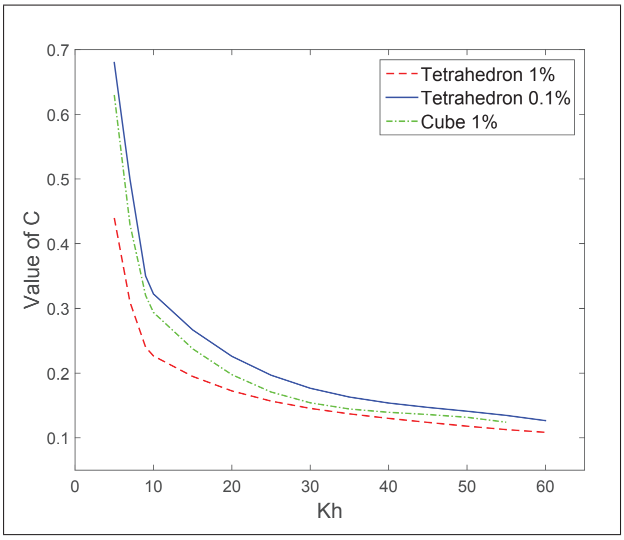

Figure 4. Coefficient C and kh_j for 1% and 0.1% error range

4. Numerical examples

The regular cube of size $2 \times 2 \times 2$ with 24 elements (14 nodes) is now chosen to simulate and analyze two acoustical problems.

4.1. Response to a prescribed velocity

For the first model, the boundary condition $\partial p/\partial n = 1$ is applied on a single triangular surface element as shown in Figure 5. Because there is no analytical solution to this problem, the quality of the solution can be assessed by looking at the imaginary part of the numerical solution since the exact solution is expected to be purely real. PUFEM results are displayed in Figure 5. The wave number k is equal to 30 so this is a relatively high frequency problem. We may note that the number of plane waves attached to each node of the cube (here Q=958) is automaticaly selected following criteria (5) as discussed in the previous section. It was checked that the magnitude of the imaginary part does not exceed 10^{-2} which is acceptable. Finally, around 15,000 dofs are used here whereas it is estimated that $1000n_\lambda^3$ dofs should be used for the same problem using classical piecewise linear or quadratic interpolation. Taking for instance $n_\lambda = 15$ degrees of freedom per wavelengths leads to an estimate of more than two millions FE nodes.

4.2. Response to a point source

The second example concerns the simulation of a sound field caused by a monopole source placed in

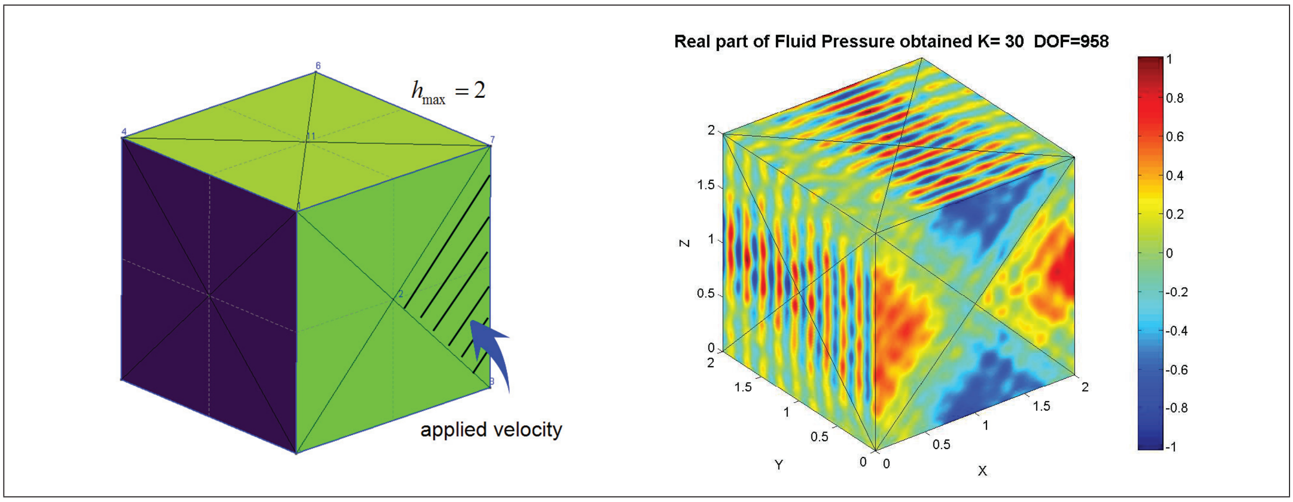

Figure 5. High frequency wave propagating problem: the model (left), PUFEM solution (right), $k = 10$ and Q=958.

a closed cavity. Here the wave equation is modified to account for the Dirac function on the right-hand side:

$$\Delta p + k^2 p = A\delta(\boldsymbol{x} - \boldsymbol{x}_0) \tag{6}$$

where A is the amplitude of the source and $\boldsymbol{x}_0$ is the source position, here we put $\boldsymbol{x}_0 = (0.9, 0.8, 1)$ which is close to the center of the cavity. The analytical solutions to this problem is available, and this will serve to measure the accuracy of the PUFEM results. Figure 6 shows some computed results at $k = 15$ using Q=278 and Q=417 plane waves directions. As expected results become more accurate using more plane waves and choosing Q=417 provides acceptable results for engineering accuracy. In this scenario, the exact solution is singular and the number of wave directions has to be chosen sufficiently high to ensure that results have converged at least at some distance from the source.

5. Conclusions

In this paper, an exact integration scheme is presented for the fast and accurate computation of highly oscillatory integrals arising from the PUFEM matrix coefficients associated with the 3D Helmholtz equation. It's shown that, through successive use of Green's theorem, volume integrals have closed-form expressions in which no integration is involved. Through convergence tests, a criteria for selecting the number of plane waves is proposed. It is shown that this number only grows quadratically with the frequency thus leading to a drastic reduction in the total number of degrees of freedoms in comparison to classical FEM. The method has been verified for two numerical examples. In both cases, the method is shown to converge to the exact solution. For the cavity problem with a monopole source located inside, the singular nature of the solution implies that more plane waves are required to attain acceptable results.

Acknowledgement

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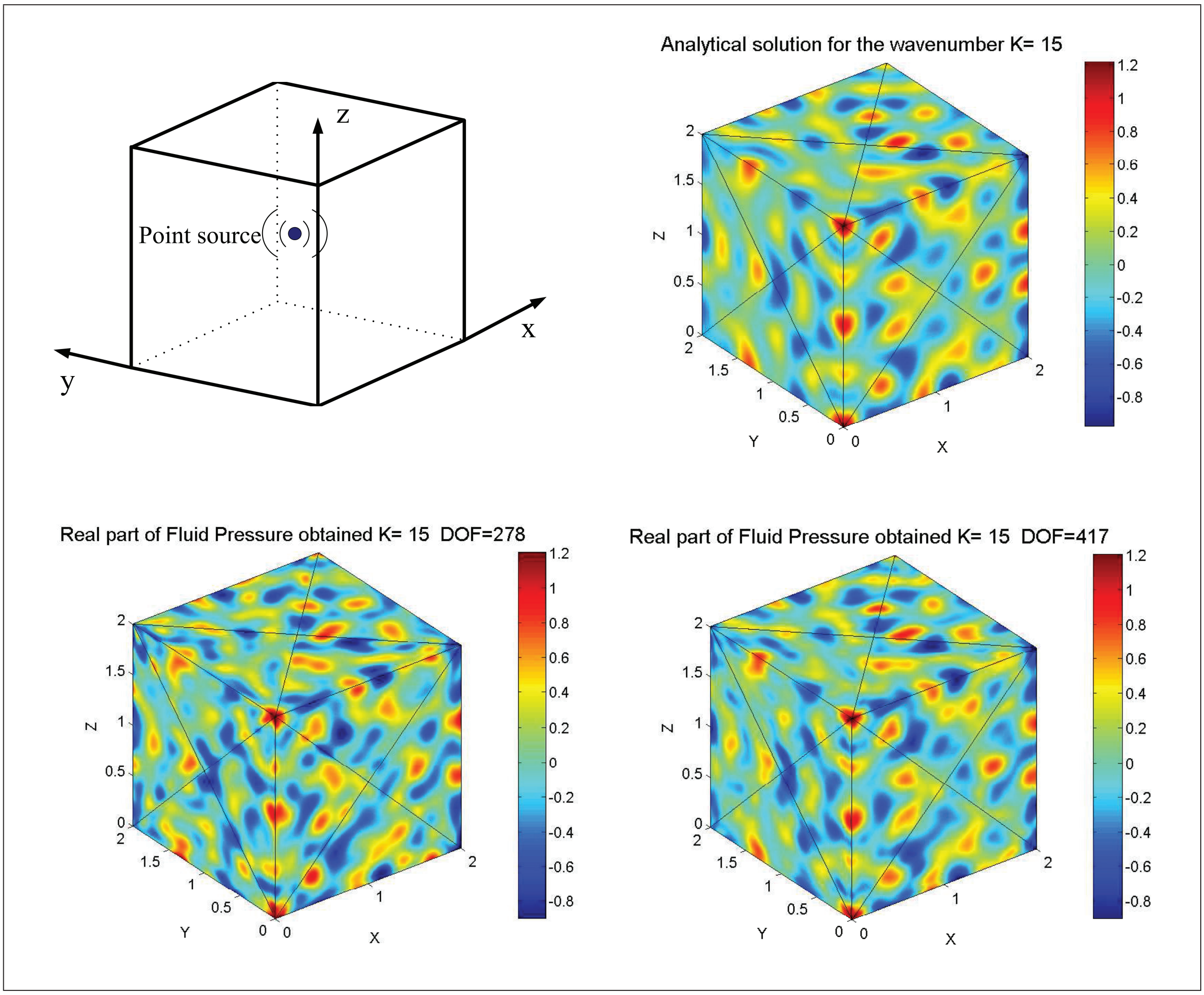

Figure 6. Pressure response in the cavity due to a point source: (a) the building model, (b) analytical solution at $k = 15$, (c) PUFEM solution with Q=278, (d) PUFEM solution with Q=417.

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