Analysis tools for multiexponential energy decay curves in room acoustics

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Introduction

The reverberation time represents one of the most important measures when describing the acoustics of a room. A well established method for calculating reverberation times is based on fitting a linear regression to a certain range of a logarithmically plotted energy decay curve (EDC). For the validity of this method, the given EDC must show a single sloped behavior before dropping below the noise level. Several deviations from this requirement have been observed since the late 1950s [1, 2], yet only one updated method for calculating the reverberation time for coupled spaces has been proposed by Xiang in [3]. However, single volume spaces can exhibit a multi exponential EDC as well.

This work deals with the search for new methods to extract decay times from a given energy decay curve, improving the commonly used linear regression approach. Three different approaches are considered, with the following three goals aiming towards a more accurate and reliable result:

- Adaption of the underlying model, such that decay curves containing multiple decay components can be evaluated.
- Robustness of the algorithm
- Minimizing computation time

Methodologies

VARPRO

The Variable Projection Algorithm (VARPRO) presented by O'Leary & Rust [4] is a revised version of the 1973 presented method by Golub and Pereyra [5]. This approach assumes that the underlying model is a linear combination of nonlinear functions. Since the introduction of this method, a wide variety of applications were found and summarised in [6]. The simplicity of using a sum of exponentials as the underlying model, its fast convergence and the possibility of implying an additional linear term are the main reasons for choosing this version in this acoustical context. Furthermore, O'Leary & Rust proposed their implementation to be a 21st century implementation of the variable projection concept using MATLAB [4].

Considering the problem of a double sloped EDC, we can describe the underlying model as

$$y(t) = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} = \eta\left(\alpha, c, t\right), \tag{1}$$

with $y(t)$ denoting the experimental data. In this special case the parameter c would appear linear, which means that for a given set of α the optimal vector c can be found using a linear least squares algorithm. Thus, defining the nonlinear problem as

$$\min_{\alpha,c} ||y - \eta\left(\alpha, c\right)||_2^2, \tag{2}$$

it can be rewritten as

$$\min_{\alpha,c} ||y - \eta\left(\alpha, c(\alpha)\right)||_2^2 \tag{3}$$

Exploiting this property, Golub and Pereyra called this a separable least squares problem and developed the variable projection method to solve it [5].

RILT

The Regularized Inverse Laplace Transform program (RILT) [7] is an emulation of the program CONTIN, which was proposed by Provencher in [8] as a general purpose constrained regularization program for inverting noisy linear algebraic and integral equations. The data obtained in various experiments represents some linear integral transform of the desired measure. In room acoustics, a measured EDC can be regarded as the Laplace transform of the present decay time distribution [1]. The inversion of such data then is an ill-posed problem with an infinite set of solutions. Thus, a standard inversion principle can not be applied and statistical regularization methods have to be used [8]. The RILT algorithm tries to find an optimal solution restricted to constraints. Prior statistical knowledge and the principle of parsimony influence the regularizor. This approach increases the accuracy by decreasing the amount of possible artefacts caused by the experimental noise [9]. Yet the inverse problem is still ill-posed and the obtained solution remains one of the infinite set of possible solutions within experimental error.

Considering the decay time distribution as the desired measure, and the EDC the indirectly obtained data, the problem can be stated as

$$y_k = \int_a^b K_k(\tau)\Phi(\tau)d\tau + n_k, \tag{4}$$

where y_k represents the measured EDC, $K_k(\tau)$ the kernel of the Laplace transform, $\Phi(\tau)$ denotes the distribution of decay times τ and n_k describes the experimental noise. After using numerical intergration to transform (4) into a system of linear algebraic equations RILT applies two different approaches for finding the "best" solution to the ill-conditioned problem:

- Constraints: By having some absolute prior knowledge about the solution a large number of possible solutions can be eliminated. Knowing, that the calculated EDC must be monotonically decreasing and concave one can restrict the decay time destribution to being non-negative.

- Regularization based on the principle of parsimony: This principle by definition searches for the simplest solution, which can be understood as looking for a solution with the least additional information to the one given by the constraints. As a result, the outcome should imply a minimal amount of artefacts.

MEDD

The Maximum Entropy Decaytime Distribution tool was derived from the Maximum Entropy Lifetime Analysis tool presented in its updated version 4.0 by Shukla in [10], which was developed to extract lifetimes from a lifetime distribution obtained in a positron lifetime experiment. Regarding an EDC as a lifetime spectrum allows the assumption of being able to adapt the given algorithm in a way that it can be used for the extraction of decay times from a given EDC.
The underlying model can be considered the same as in (4), thus the matrix representation resolves as

$$Y = K\Phi + N. \tag{5}$$

To approach this problem, the MELT algorithm uses a quantified maximum entropy method. The Maximum Entropy Principle, first proposed in [11] as a natural connection between information theory and statistical mechanics, yields the opportunity of calculating the least biased estimate for a problem based on the given information. This method allows to find an estimate of a positive additive distribution (PAD) from noisy and incomplete data based on a Bayesian framework. Knowing a number of various solutions A, B, C one can then describe them as conditional probabilities pr(A|D), pr(B|D), pr(C|D). So if Φ represents a particular solution, one needs the probability distribution pr(Φ |D) subject to Φ. This is not directly obtainable from the given dataset D. However, the reversed conditioning pr(D| Φ) is, which is better known as the "likelihood". Assuming the experimental noise to be uncorrelated and Gaussian, the probability density $p_G(N)$ with the noise being described as $N = D - K\Phi$ would resolve as

$$\begin{aligned} pr(D|\Phi) &= p_G(N) = p_G(D - K\Phi) = \\ &= \prod_{j=1}^{N_{dat}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{1}{2\sigma^2}\left[d_j - \sum_{\mu}^{N_{mod}} k_{j\mu}\phi_\mu\right]^2\right), \end{aligned} \tag{6}$$

the connection to the desired probability can be found using Bayes theorem

$$pr(\Phi|D) = \frac{pr(\Phi)pr(D|\Phi)}{pr(D)}, \tag{7}$$

with pr(D) being a normalising factor to ensure that the sum of the probabilities of all possible solutions is equal to one and, therefore, assuring the presence of a probability density. Considering the pr(D|Φ) as given and pr(D) being a normalising constant, the only remaining unknown term is pr(Φ), which stands for the "prior probability" distribution of Φ. According to [12] the pointwise probability

$$p(\phi|m, \alpha) \propto \exp\left(\alpha S(\phi, m)\right) \tag{8}$$

reflects the most important part of the quantified entropic prior. The two parameters m and α represent a model for ϕ and an inverse measure of the expected spread of values of ϕ about m. The function $S(\phi, m)$ denotes the Shannon-Jaynes entropy [11]. Thus, the pointwise joint probability distribution is

$$\begin{aligned} pr(\phi, D|m, \alpha) = \left(\frac{\alpha}{2\pi}\right)^{\frac{r}{2}} \\ \prod \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\alpha S(\phi, m) - \frac{1}{2}C(\phi, D)\right). \end{aligned} \tag{9}$$

Since the Shannon-Jaynes entropy is a convex function with negative definite curvature and $C(f)$ positive, the posterior probability $pr(\phi|m, \alpha, D)$ has a unique maximum at

$$\alpha\frac{\partial S}{\partial \phi} - \frac{1}{2}\frac{\partial C}{\partial \phi} = 0 \quad \text{at} \quad \phi = \hat{\phi} \tag{10}$$

The obtained distribution $\hat{\phi}$ then is the single most probable PAD. The choice of a well guessed kick-off solution is essential as it has a stabilizing and regularizing effect. Furthermore it shortens the time needed to converge. For this particular reason the algorithm uses a general optimal linear filter [13] to compute a good kick-off solution.

Experimental Results

To evaluate the results obtained using the different algorithms, a experimental data set from a reverberation room measurement was selected. As a representative the $f = 500$Hz octave band has been chosen. The resulting fitting curves as well as the calculated residuals are presented in Fig.1. Clearly visible, the residuals within the time range of 0s to 0.4s can be linked to the fluctuation in the EDC, therefore the progression of the residuals as a function of time for the different algorithms follows a similar behavior. In the later part (0.4s - 1.3s), only the trend of the VARPRO and the RILT algorithms match. The MEDD algorithm for this section achieves a better result, which is visible in the fitting curves as well as the residuals. As a measure for the accuracy of the obtained fitting curves, the sum of residuals is presented in Tab.1, aside the calculated reverberation times and their corresponding intensities.
Figure 2 shows the obtained decay time distribution for the three different algorithms. As described in the methodologies section, the VARPRO and the RILT algorithm result in discrete decay times, thus, the resulting "distribution" for the VARPRO algorithm only contains the two evaluated decay times. The RILT distribution

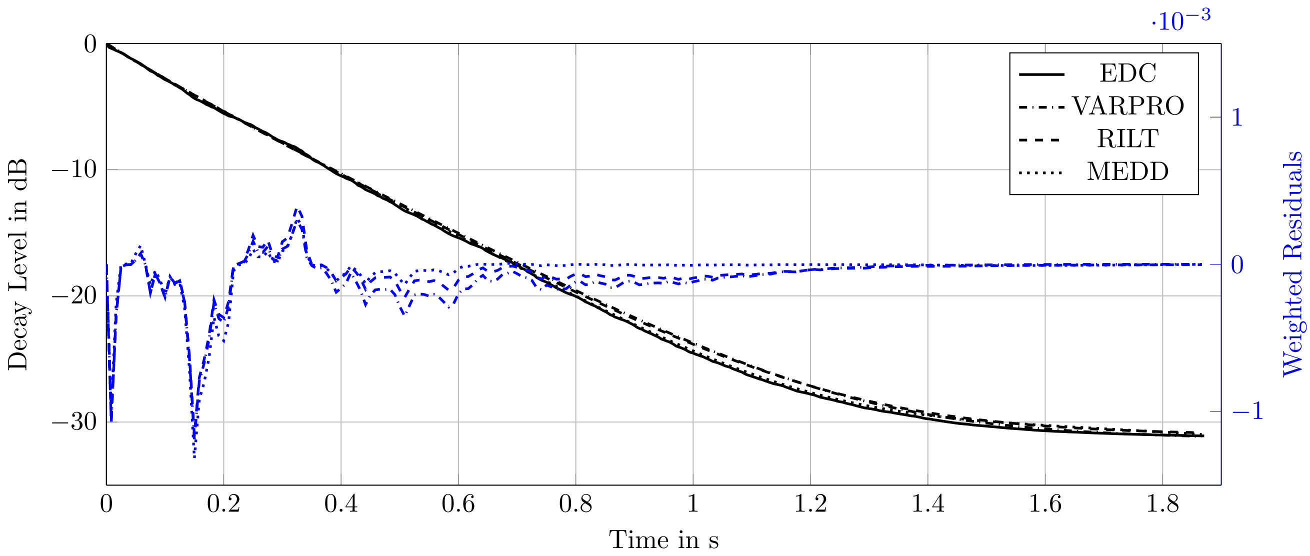

Figure 1: Comparison of (black) the measured and estimated EDCs and (blue) their corresponding weighted residuals [Measurement: 500Hz, 20 diffusors, 10.8m^2 porous absorber]

represents the predefined decay time grid and the corresponding evaluated intensities. With MEDD a continuous decay time distribution is obtained. In all three computed distributions, the second peak is higher. This results represents the fact that the time range, where the initial decay time T_1 is dominant is shorter. The slightly higher estimated T_2, obtained using VARPRO and RILT also reflects the deviation from the measured EDC during the time interval of 0.4s up to 1.4s.

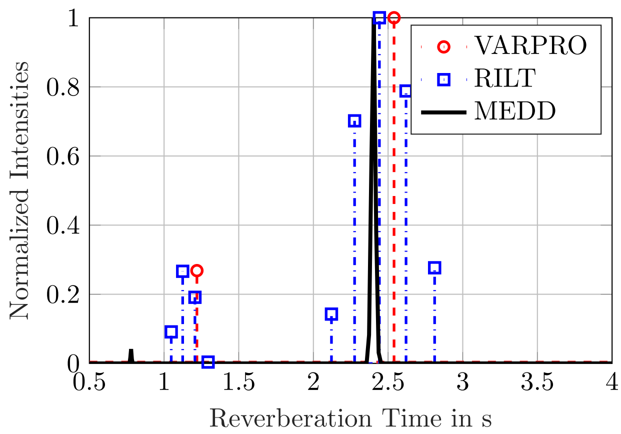

Figure 2: Comparison of the computed reverberation time distributions [Measurement: 500Hz, 20 diffusors, 10.8m^2 porous absorber]

	VARPRO	RILT	MEDD
T_1 in s	1.22	1.12	0.78
Int of T_1	0.20	0.21	0.11
T_2 in s	2.54	2.44	2.40
Int of T_2	0.80	0.79	0.89
$\sum Res$	0.0477	0.0390	0.0264

Table 1: Reverberation Times T_1 & T_2, their corresponding intensities and the sum of residuals

Considering the measurement from a reverberation chamber, the number of peaks present in each octave band was computed using MEDD. Figure 3 shows a comparison of the number of slopes present in an empty reverberation chamber. For each octave band the peaks were computed and plotted neglecting their corresponding intensity. The x-axis represents the decaytime grid used by MEDD. The corresponding reverberation time can be computed using

$$\tau = \exp\left(const + \frac{i}{increment}\right) \cdot \frac{13.8 \cdot down}{fs}, \quad (11)$$

where $const = 2$, $i = 1 : 1000$ and $increment = 200$ represent decay time grid variables defined for the computation process, $down = 100$ and $fs = 48000$ represent the downsampling factor and the sampling frequency respectively. This method allows to easily visualize the number of slopes present in a certain frequency band. For the chosen example, no absorbing specimen and no diffusors were place inside the reverberation chamber. In none of the evaluated frequency bands (125Hz to 4kHz) a single sloped EDC could be detected, and thus the requirement of having a single slope for deploying the linear regression method is not fullfilled. Considering the introduction of an absorbing specimen, this effect will be increased due to the non-uniform distribution of absorption in the room. Further, the expected trend towards lower reverberation times at higher frequencies is visible.

Discussion

The adapted version of the VARPRO algorithm offers a time efficient alternative to fitting a linear regression to the data. Previous to triggering the calculation process, the number of slopes present in the given decay curve has to be entered. This fact and the loss of the robustness for more than one decay component make this algorithm a weak choice. If only one exponential term is fitted to the data, the solution represents an accurate fit of the first

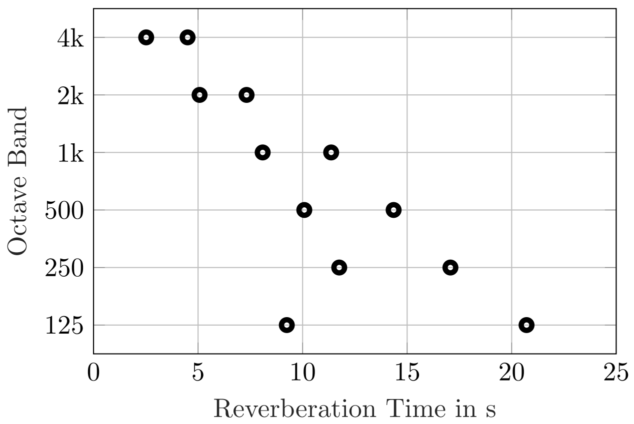

Figure 3: Number of peaks/slopes present in a MEDD evaluation of a reverberation chamber measurement without absorbing specimen and diffusing elements [Measurement: 500Hz, 0 diffusors, no absorbing specimen]

slope. As a result, VARPRO can be used as a tool to compute an estimate for T_{early} from a given EDC. Considering the weaknesses of the VARPRO algorithm, the second adapted algorithm offers the possibility to fit a sum of exponentials to the data, without the handicap of having to know the number of present slopes.
The RILT algorithm offers a great tool to obtain the corresponding intensities for a given decay time grid using a nonlinear least squares fitting approach. The computation time increases exponentially with the number of grid points which decreases its potential when investigating a wider decay time distribution. Additionally, the obtained intensities as a function of the decay times can be regarded as a decay time distribution.
The results gained using MEDD underline the expectation of the existence of multiple sloped decay curves in a reverberation chamber. More than one peak in the obtained decay time distributions is visible for most of the investigated datasets. The novel method for extracting decay times from a given measured EDC in room acoustics represents a time-efficient tool to evaluate more than a single slope. Furthermore, the results obtained by MEDD could be used for the calculation of the absorption coefficient and the development of a new measure for the multiple slope effect.

Conclusion

This work presents a contemporary study of algorithms to analyze the decay times present in a given EDC. Based on the assumption of the EDC being the Laplace transform of the decay time distribution, the MEDD algorithm was implemented to use a quantified maximum entropy method for estimating the inverse Laplace transform from the EDC, yielding the decay time distribution. The results showed, that for a measurement done in a reverberation chamber, the EDC in most cases exhibits a multiple sloped behavior.

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