Explosive instabilities in antiferromagnetic crystals due to the 3-phonon interaction mechanism

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It is well known that periodic modulation of sound velocity in a material can lead to an exponential increase in amplitude (parametric instability). At the same time, periodic modulation of a nonlinear material parameter can result in an even more powerful instability having the character of a mathematical singularity. In the phonon representation, this mechanism corresponds to the three-phonon interaction instead of interaction of two phonons in an ordinary parametric process. Here we suggest a system in which the explosive scenario is theoretically described. Our case study concerns magnetoelasic Lamb waves in a magnetic plate subject to the electromagnetic modulation of the third order elastic modulus. In addition to the Lamb wave and the modulated magnetic field, a shear resonant mode is excited in the plate. We show that even a weak Lamb wave launched in such a plate will induce a backward propagating Lamb wave, with amplitudes of the both waves rapidly increasing. At the same time, the shear resonance amplitude is also enormously boosted, so that all three phonons (two of the Lamb waves and one of the shear mode) experience a sort of a positive feedback. The process requires the fulfillment of certain resonance conditions easily interpreted as a phonon interaction diagram. However, even if the magnetic pumping is never depleted, there exist factors limiting the amplitude in practice. Here we study a mechanisms based on the fact that resonant frequencies depends on amplitudes in a nonlinear material thus introducing a resonance detuning that increases with growing amplitude. As a result, the mathematical singularity in the time dependency of the amplitudes does not appear, whereas the amplification factors still remains huge.

1 Introduction

Instabilities in dynamic systems attract the attention of researchers due their significant positive and negative impacts. On one hand, development of instabilities efficiently generates noise, vibrations, etc., finally resulting in fatigue, wear, damage of components and structures. On the other hand, instabilities generated in a controlled environment can have specific applications. For instance, an effective conversion of external pumping energy into mechanical energy offers an opportunity to use the effect for creating micro-actuators. Besides, giant amplification of weak signals can help design hyper-sensitive miniaturized sensors. These factors motivate our interest to establishing the conditions of the huge amplitude growth effect and to finding exemplar systems in which these conditions can be theoretically predicted and experimentally observed.

There are at least two confirmed types of behavior characterized by a theoretically infinite amplitude growth: exponential and explosive. The exponential instability appears when a linear parameter such as stiffness in oscillators or sound velocity in acoustics is efficiently modulated by another physical process. This effect is usually called parametric amplification and is typical for a wide range of situations ranging from the classical pendulum with a variable string length to stimulated processes in laser physics [1], light scattering [2,3], acoustics [4], etc.

Our interest here is to another type of growing instabilities having the explosive behavior. In this case, an external process modulates not the linear parameter but the quadratic nonlinear coefficient. The difference between the "usual" parametric instability having the exponential character and the explosive effect of nonlinearity modulation can be understood using the Hamiltonian formalism. The classical Hamiltonian contains terms with two multiplied amplitudes in the former case and with three amplitudes in the latter case. Application of the appropriate resonance conditions produces terms containing a combination of two or three complex conjugate amplitudes, respectively. In a quantum description of such interaction between two or three phonons, two or three creation operators appear. The presence of the third creation operator explains an additional contribution to the amplification process and results in an explosive amplitude growth when theoretically infinite values are obtained at a finite moment of time, as it is for the mathematical singularity.

However, in reality, there exist several factors limiting the infinite amplitude growth. One of them is amplitude dependency of the resonant frequency that creates a detuning from resonance limiting the amplitude.

So far explosive instability has been experimentally observed in plasma [5,6]. Here we consider another system in which the effect is theoretically expected. We numerically demonstrate the amplitude growth with the vertical asymptote, then introduce the nonlinear frequency detuning factor, and analyze its influence.

2 Equations for explosive instability in an antiferromagnetic plate

The objective of this section is to derive equations describing the explosive instability in a particular system where the three-phonon interaction appears. The system represents an antiferromagnetic plate in which a Lamb wave propagates in the presence of pumping realized by means of an alternative magnetic field. In addition, a shear standing wave is to be generated. In this situation, another Lamb wave with the opposite propagation direction is spontaneously excited. The three phonons necessary for the explosive instability generation are coming from the two Lamb waves and from the shear resonance mode. The magnetic pumping action modulates the quadratic nonlinear parameter and actually provides energy for the explosive amplitude growth.

As a model medium we choose an antiferromagnetic crystal with the magnetic anisotropy of the "easy plane" type belonging to symmetry group D_{3d}^{6} (e.g. α-Fe_2O_3 or $FeBO_3$). The crystal has a shape of a plate cut in the basal plane normal to the crystallographic axis $C_3 \parallel z$ (see Figure 1). We suppose that the plate is placed in a constant magnetic field $\vec{H}$ parallel to y-axis and in a transversal RF magnetic field $\vec{h}_p(t)$ parallel to the binary axis $U_2 \parallel x$ (see Figure 1). The instability effect is produced by the interaction of the fundamental shear mode with the in-plane displacements parallel to the binary axis x and two asymmetric Lamb waves with polarization normal to the plane and with the wave vectors $\pm k$ parallel and antiparallel to the x-axis.

It is possible to show [7] that the potential energy density in the material has the form:

$$F = 2C_{44}u_{xz}^2 + \Psi_p h_p(t) u_{xz}^3 , \quad (1)$$

where ρ is density of the crystal, C_{44} is the shear elastic modulus and Ψ_p is the amplitude of interaction caused by modulation of the nonlinear elastic parameter $C_{555}\left(\vec{H}\right)$:

$$\Psi_p = \frac{1}{3}\frac{\partial}{\partial H_x} C_{555}\left(\vec{H}\right), \quad (2)$$

An explicit expression for Ψ_p applicable to the antiferromagnetic with the easy type magnetic anisotropy of D_{3d}^6 symmetry in transversal alternative magnetic field is derived in [8]. In the particular case when the only nonzero strain component is u_{xz}, Ψ_p equals to

$$\Psi_p = -16 C_{44}\zeta^4 \frac{1}{\varepsilon}\frac{H+H_D}{\left(\omega_{s0}/\gamma\right)^2}\Xi , \quad (3)$$

where

$$\Xi = 1 - \frac{H_D H_E H_{ms}}{2\left(H+H_D\right)\left(\omega_{s0}/\gamma\right)^2} , \quad (4)$$

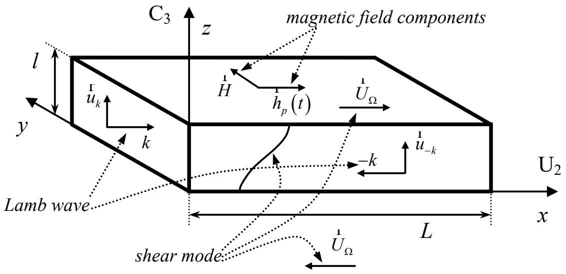

Figure 1: System's geometry. Wave displacements $\vec{u}_k$ and $\vec{u}_{-k}$ for the Lamb waves with wave vectors $\vec{k}$ and $-\vec{k}$ are shown as well as wave displacement $\vec{U}_\Omega$ for the shear mode. Magnetic fields $\vec{H}$ and $\vec{h}_p(t)$ are also plotted.

In Eqs. (3)-(4), $\varepsilon=2B_{14}/C_{44}$ is the spontaneous magnetostrictive strain, B_{14} is a magnetoelastic constant, H_E, H_D and H_{ms} are exchange, Dzyaloshinsky and magnetoelastic effective fields, respectively, ω_{s0} is the frequency of antiferromagnetic resonance, γ is the magneto-mechanical ratio, ζ is the magnetoelastic coupling coefficient. The details of this derivation can be found in [7].

In Eq. (1), the pumping magnetic field that modulates the quadratic nonlinearity coefficient is chosen as

$$h_p(t) = h_0 e^{i\omega_p t} + c.c., \quad (5)$$

where ω_p is pumping frequency, h_0 is the magnetic field amplitude.

The displacement field is assumed to have the following structure:

$$u_x = (De^{i\Omega t} + D^* e^{-i\Omega t})\cos\left(\frac{\pi}{l}z\right), \quad (6)$$

$$u_z = \left(Ae^{-ikx} + Be^{ikx}\right)e^{i\omega_k t}\sin\left(\frac{\pi}{l}z\right), \quad (7)$$

Here the contribution u_x corresponds to the shear resonance mode with the frequency Ω and amplitude D, while u_z-component describes the Lamb waves with the correspondent wave number k and frequency ω_k. The Lamb waves have approximately vertical displacement since they are considered in the short-wave approximation in order to make use of the fact that wave interactions enhance when the wavelength decreases. Amplitude A of the forward wave is coming from the excitation signal while the backward wave of the amplitude B is not deliberately excited but appears spontaneously as it will be demonstrated. In Eqs. (6)-(7), l is the plate thickness.

The equations of motion corresponding to the potential energy density Eq.(1) have the form:

$$\rho\frac{\partial^2 u_x}{\partial t^2} = C_{44}\frac{\partial^2 u_x}{\partial z^2} + \frac{3}{2}\Psi_p h_p(t) u_{xz}\left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial x}\right), (8)$$

$$\rho\frac{\partial^2 u_z}{\partial t^2} = C_{44}\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_x}{\partial x \partial z}\right) + \frac{3}{2}\Psi_p h_p(t) u_{xz}\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z \partial x}\right) (9)$$

Equations for amplitudes are obtained from Esq. (8)-(9) in the following way. First, Eq. (8) is multiplied by $\cos\left(\pi z/l\right)$, Eq. (9) is multiplied by $\sin\left(\pi z/l\right)$, and both equations are integrated over the plate thickness $0 \le z \le l$. Then two resulting equations are obtained, into which the explicit forms Eq. (6)-(7) have to be substituted. Since amplitudes A, B, and D evolve slowly in comparison to fast terms with frequencies ω_k, ω_p, and Ω, their double derivatives can be neglected. Finally, only resonant terms with

$$\omega_p - 2\omega_k - \Omega \approx 0 , \quad (10)$$

should be retained. The eventual result for the slowly varying amplitudes is presented in the form of the following equations:

$$\frac{\partial A}{\partial t} + v\frac{\partial A}{\partial x} + \delta_1 A = -\frac{i}{\rho\omega_k l}\Psi_p h_0 k^2 D^* B^* e^{i\left(\omega_p - 2\omega_k - \Omega\right)t}, (11)$$

$$\frac{\partial B}{\partial t}-v\frac{\partial B}{\partial x}+\delta_1 B=\frac{i}{\rho\omega_k l}\Psi_p h_0 k^2 D^* A^* e^{i(\omega_p-2\omega_k-\Omega)t},\quad(12)$$

$$\frac{\partial D}{\partial t}+\delta_2(D-D_0)=-\frac{i}{\rho\Omega l}\Psi_p h_0 k^2\frac{1}{L}\int_0^L A^* B^* e^{i(\omega_p-2\omega_k-\Omega)t},\quad(13)$$

where damping factors δ_1 and δ_2 have been additionally introduced. Here v is the group velocity of the Lamb waves.

Here it is appropriate to mention that an attempt to build up the classical Hamiltonian corresponding to Eqs. (11)-(13) will produce a term containing $h_0 e^{i\omega_p t}\left(d^*+d\right)a^*b^*+c.c.$, where a, b, and d are the canonical variables corresponding to amplitudes A, B, and D, respectively. The combination $d^*a^*b^*$ has a quantum counterpart in the form of multiplication of three phonon creation operators. This fact indirectly explains the explosive growth effect.

3 Numerical demonstration of the explosive instability

For the numerical analysis, it is convenient to rewrite Eqs. (11)-(13) in the following form:

$$\frac{\partial A}{\partial t}+\frac{\partial A}{\partial x}+\delta_1 A=-i\Phi B^* D^*,\quad(14)$$

$$\frac{\partial B^*}{\partial t}-\frac{\partial B^*}{\partial x}+\delta_1 B^*=i\Phi AD,\quad(15)$$

$$\frac{\partial D}{\partial t}+\delta_2\left(D-D_0\right)=-i\mu\frac{1}{L}\int_0^L dxA^*B^*,\quad(16)$$

Here time t is measured in microseconds, x and L are normalized on the group velocity v, new amplitudes A and B are obtained by adding a factor k/ε, amplitude D is multiplied by $\pi l\varepsilon/2$ (ε is the spontaneous magnetostrictive strain introduced above), detuning from resonance $\Delta\omega_p=\omega_p-2\omega_{k_0}-\Omega$ is neglected, the interaction amplitude Φ is defined as

$$\Phi=\frac{2k\varepsilon}{\pi\rho v_g}\Psi_p h_0,\quad(17)$$

and a new parameter $\mu=\Phi\Omega/\left(8\omega_k\right)$ is introduced. Here parameter D_0 corresponds to a continuous excitation of the resonance mode by an external alternative force. Basically, in experiments such force is created by an additional alternative magnetic field applied at the eigenfrequency of the mode [7,9].

Equations (14)-(16) are to be completed by the boundary and initial conditions:

$$A\big|_{x=0}=A_0\left(t\right),\ A\big|_{t=0}=0,\quad(18)$$

$$B\big|_{x=L}=0,\ B\big|_{t=0}=0,\quad(19)$$

$$D\big|_{t=0}=D_0,\quad(20)$$

where $A_0(t)$ is the amplitude of an incident wave at the entrance $x=0$ to the active zone.

Equations (14) and (15) describe the parametric phase conjugation of traveling waves as the presence of complex conjugate amplitudes in the right-hand sides. These conjugate amplitudes contribute into Eqs. (14) and (15) together with the shear excitation D and variable Φ corresponding to the pumping magnetic field (see Eq. (17)). At the same time, Eq. (16) introduces a feedback effect into the system, when the signal (traveling Lamb waves) impacts the pumping (shear resonance). In the absence of the feedback effect, the amplitudes of Lamb waves would exponentially increase [10,11] once the threshold of parametric instability is reached. As we will show here, the addition of feedback in Eq. (16) considerably modifies the behavior of the system. Due to the feedback, the exponential amplification scenario is followed by the explosive instability.

Accepting the following typical values of physical parameters of the problem: $\omega_k/(2\pi)$=20 MHz, $\Omega/(2\pi)$=1 MHz, acoustical quality factor of 10^3, $v=10^5$ m/s, L= 4 cm, H=0.5 kOe, h_0=40 Oe, and magnetic parameters for the antiferromagnetic crystal taken from [7,12], we obtain the normalized parameters $\delta_1=6\cdot10^{-2}$ (μs)$^{-1}$, $\delta_2=3\cdot10^{-3}$ (μs)$^{-1}$, L=10 μs, Φ=10 (μs)$^{-1}$, $\mu=6.25\cdot10^{-2}$ (μs)$^{-1}$ in Eqs. (14)-(16). In Figure 2 below, t, x, and L are measured in microseconds.

In the boundary condition Eq. (18), an explicit form for $A_0(t)$ should be set. In fact, in the situation of the giant amplification considered here the exact shape of the "starter" signal is not essential. We choose a Gaussian pulse

$$A_0\left(t\right)=A_0\exp\left(-\frac{\left(t-t_0\right)^2}{2w^2}\right),\quad(21)$$

of duration w=0.5 μs centered at t_0=2 μs. Two remaining parameters, A_0 and D_0, determining the boundary conditions Eqs. (18)-(20) are already normalized on the spontaneous magnetostrictive strain $\varepsilon\approx10^{-5}$. Therefore $A_0=10^{-2}$ taken here as an example corresponds to a low strain of about 10^{-7}. The shear mode amplitude D_0 plays the pole of a pumping; a chosen value $D_0=5\cdot10^{-2}$ actually means that the considered pumping amplitude is quite low (about $5\cdot10^{-7}$) and can be increased at least by a factor of 10^1-10^2. The normalized amplitudes can reach values of order of 10^2 (physical strains about 10^{-3}); at higher strains the crystal fails.

This numerical example shows that in the considered regime the starting values of signal or shear pumping have little influence. Sooner or later, the explosive instability develops with a vertical asymptote while the parametric process (no shear wave feedback) results in the exponential amplitude growth (linear in the logarithmic scale).

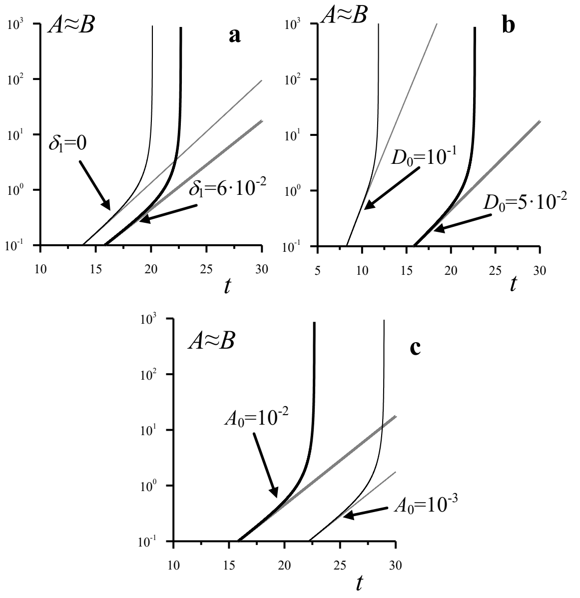

Figure 2: Time dependencies for the amplitudes $A \approx B$ at the centre of the plate i.e. at $x=L/2$ showing explosive (black curves) and exponential (gray curves) instabilities. The former case occurs when the additional resonant shear mode pumping is applied while the latter situation corresponds to the classical parametric interaction (no additional shear action, magnetic pumping only). The vertical axis is shown in the logarithmic scale. Sets (a)-(c) illustrate the process at different values of parameters δ_1, D_0, and A_0, respectively. The baseline curves (thick lines) are the same in all the three sets.

4 Effect of the nonlinear frequency shift

However, in reality, rapidly growing amplitudes are always limited by a number of factors. The most obvious of them is pump depletion; indeed, infinite amplitudes are not possible at least since the pumping can not provide infinite energy. Other processes limiting the amplitude nonlinear frequency shift due to strong cubic nonlinearity in an antiferromagnetic crystals. Taking into account this mechanism characterized by coefficient R, we arrive at a system of corrected equations:

$$\frac{\partial A}{\partial t}+\frac{\partial A}{\partial x}+\delta_1 A-i\omega_k R\left(2|D|^2-2|D_0|^2+2|B|^2+|A|^2\right)A=-i\Phi B^* D^*, \quad (22)$$

$$\frac{\partial B^*}{\partial t}-\frac{\partial B^*}{\partial x}+\delta_1 B^*+i\omega_k R\left(2|D|^2-2|D_0|^2+2|A|^2+|B|^2\right)B^*=i\Phi AD, \quad (23)$$

$$\frac{\partial D}{\partial t}+\delta_2\left(D-D_0\right)-i\Omega R\left(|D|^2-|D_0|^2+\frac{2}{L}\int_0^L\left(|A|^2+|B|^2\right)dx\right)= \quad (24)$$
$$=-i\mu\frac{1}{L}\int_0^L dx A^* B^*.$$

The corresponding solution is depicted in Figure 3. If the frequency shift factor is taken into account, the initial rapid amplitude growth is followed by an oscillatory behavior with an increasing frequency of oscillations. The effect can be interpreted in the following way. The nonlinear frequency shift breaks the synchronism between the generated waves and the pumping, therefore the amplitudes decrease. This decrease partly restores the synchronism which results in the subsequent amplification growth. Finally, depending on the initial amplification increment, one of two situations can occur: a trend towards a stationary level with lowering frequency of amplitude oscillations or persisting amplification with increasing frequency of amplitude oscillations. The latter case is illustrated in Figure 3. The calculations are terminated at t=27 μs in our example since after due to rapid increase of the vibration frequency they become less reliable or require a larger number of discretization points.

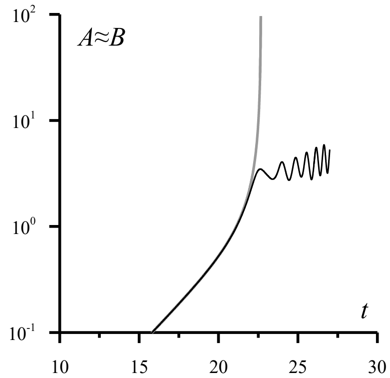

Figure 3: Time dependencies for the amplitudes $A \approx B$ at the centre of the plate in the case with (thin black line) and without (thick gray line) resonant frequency detuning.

However, it is important to mention that, despite of the amplitude limit effect of the nonlinear frequency shift effect, the amplitude growth remains high and reaches a factor of 10^3 or more for smaller starter signal amplitudes.

6 Conclusions

The analysis and numerical examples we present are related to systems with two-and three-phonon interactions. Two-phonon processes described here correspond to the classical parametric interaction of the kind $\omega_p=\omega_k+\omega_{-k}$, where the pumping wave of frequency ω_p exponentially amplifies signals at frequencies ω_k and ω_{-k}. In the considered case, Lamb waves of frequencies ω_k and ω_{-k} propagate in a plate made of antiferromagnetic material in which a transverse alternative magnetic field of frequency ω_p is applied. The situation changes considerably if an additional pumping channel is introduced in the form of a shear resonant mode of frequency Ω. The corresponding three-phonon process $\omega_p=\omega_k+\omega_{-k}+\Omega$ generates instabilities of much more "powerful" (explosive) type when time dependencies of signal amplitudes behave as a mathematical singularity. This offers an opportunity to convert the magnetic energy into mechanical energy in an extremely efficient manner.

We also analyze one of factors limiting the infinite amplitude growth in practice, such as resonant frequency detuning due to nonlinear amplitude dependency of sound velocity in a material. However, even with the account for this limiting factor, the amplitude increase remains huge.

An antiferromagnetic crystal excited in a way described here is only an example of a situation in which the derived explosive instability equations are applicable and the three-phonon interaction takes place. The considered nonlinearity modulation mechanism is possible to extend on systems of different physical nature and to apply in acousto-electronics, electro- and hydrodynamics and in microsystems designing.

Acknowledgements

The work was carried out in the framework of the Russian Federation program (AAAA-A18-118021390174-1) and supported by the Presidium of the Russian Academy of Sciences (Program I.7) and by CNRS funding.

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