PIEZOELECTRIC VIBRATING BEAM RESONATORS REVISITED: FLEXURAL AND TORSIONAL MODES OF GALLIUM ORTHOPHOSPHATE GaPO₄

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Abstract

Flexural modes of Quartz crystal beams are widely used for wrist watch resonators and vibrating beam resonators for sensors. Torsional modes of quartz have also been studied even if less applications have been developed. We present here a first evaluation of both flexural and torsional modes of rectangular cross-section vibrating bars of Gallium Orthophosphate (GaPO₄). An analytical model of vibrating beam resonators in flexural and torsional modes is built and validated by comparison with Quartz data on similar structures. Temperature effects are introduced by the approximate but classical method of varying effective material constants, beam dimensions and crystal mass density versus temperature. Temperature-compensated GaPO₄ orientations of vibrating beam in torsional modes are predicted.

Introduction

The piezoelectric crystal of gallium orthophosphate (GaPO₄) is a new quartz homeotypic crystal. Due to the lack of the α−β phase transition most material properties are stable up 970°C. Its excellent thermal stability opens new field of applications for high temperature sensors. Flexural modes of quartz crystal beams are widely used for wrist watch resonators and vibrating beam resonators for sensors. Torsional modes of quartz have also been studied even if less applications have been developed.

This paper reports a first evaluation of both flexural and torsional modes for rectangular cross-section vibrating beams of GaPO₄. An analytical model of vibrating beam resonators in flexural and torsional modes is built and validated by comparison with quartz data on similar structures.

Temperature effects are introduced by the approximate but classical method of varying effective material constants, beam dimensions and crystal mass density versus temperature. Temperature-compensated cut for GaPO₄ in vibrating beam in torsional modes are predicted.

I. Analytical model of flexural modes

![Figure 1: Geometry of beam in flexural modes](image)

The geometry of the rectangular beam is represented in figure 1. As a first approach, the analytical model of flexural modes is built without taking into account the piezoelectric effects and mass loading [1]. Thus, the equation of motion (1) for this vibrating beam model is:

\[
\frac{1}{s_{22}} I \frac{d^4 w}{d y^4} - \rho A \frac{d^2 w}{d t^2} = 0 \tag{1}
\]

where:
- \( I \): Inertia of the beam
- \( A \): Area of the beam's section
- \( \rho \): Material's density
- \( s_{22} \): Compliance constant for Y axis
- (x, y, z): Particules coordinates
- u, v, w: Mechanical displacements
- X, Y, Z: Axes

With (1), the resonant frequencies (2) are determined:

\[
f = \frac{\lambda^2}{2 \pi L^2} \sqrt{\frac{I}{\rho A s_{22}}} \quad \text{with} \quad \lambda = \alpha L \tag{2}
\]

\[ L \]: Length of the beam

\[ \alpha = \frac{\rho S_{22} A \omega^2}{I} \]

However, the GaPO₄ is an anisotropic material then these properties depend on the orientation. Therefore the frequency depends on the cut angle [1][2]. In this study, we use only the rotated X-cut
II. Temperature effects of flexural modes

The dependance of the frequency versus the temperature is an essential characteristic to resonators or sensors which control the frequency.

The dependence of resonant frequencies on temperature is defined in the equation (3):

\[ f(T) = f(T_0)(\alpha(T-T_0) + \beta(T-T_0)^2 + \gamma(T-T_0)^3) \]  

with 
\[ \alpha = \frac{1}{f} \frac{\partial f}{\partial T} ; \beta = \frac{1}{2!} \frac{\partial^2 f}{\partial T^2} ; \gamma = \frac{1}{3!} \frac{\partial^3 f}{\partial T^3} \]

The first three Temperature Coefficients of Frequency (TCF)

The Studies of temperature effects on vibrating beam resonator in flexural modes are made in [4]. Contrary to quartz, the 1st TCF of GaPO₄ has two zero values for particular angles \( \theta = -16.7^\circ \) and \( \theta = -48^\circ \). Hence compensation of temperature effects to the first order exists. Temperature compensated cuts for flexural modes are very else to what is found for lengthextensional modes.

III. Analytical model of torsional modes

The other vibrating beam resonator studied is vibrating beam in torsional modes [5][6]. The figure 3 shows the geometry of the rectangular cross-section of the beam in torsional modes.

The mechanical displacements \( u, v, w \) along the axes X, Y, Z are:

\[
\begin{align*}
\begin{cases}
  u(y,t) = -\tau y z \\
  v(y,t) = \tau \phi(x,z) \\
  w(y,t) = \tau x y
\end{cases}
\end{align*}
\]  

where \( \phi(x,z) \) is the St Venant’s torsion function and \( \tau = \frac{\partial \alpha}{\partial y} \) is the torsional angle per unit length with \( \alpha \) rotation angle.

The strains in the beam are \( S_4 \) and \( S_6 \) and the corresponding stresses, assuming \( \tau \) constant are:

\[
\begin{align*}
T_4 &= \tau \left[ C_{44} \left( \frac{\partial \phi}{\partial z} + x \right) + C_{46} \left( \frac{\partial \phi}{\partial x} - z \right) \right] \\
T_6 &= \tau \left[ C_{46} \left( \frac{\partial \phi}{\partial z} + x \right) + C_{66} \left( \frac{\partial \phi}{\partial x} - z \right) \right]
\end{align*}
\]  

In the rotated X-cut, the elastic constant \( C_{46} \) is equal to zero, then the function \( \phi(x,z) \) is given by (6):

\[
\phi(x,z) = -xz + \frac{8z^2}{\pi^4} \sqrt{\frac{C_{66}}{C_{44}}} \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{(2n+1)^3} \right\} \sinh \left( \frac{(2n+1)\pi x}{2z_o} \right) \left( \frac{C_{44}}{C_{66}} \right) \cosh \left( \frac{(2n+1)\pi x}{2z_o} \right) \left( \frac{C_{44}}{C_{66}} \right) \sin \left( \frac{(2n+1)\pi z}{2z_o} \right) \left( \frac{C_{44}}{C_{66}} \right) \quad (6)
\]

The torque \( M_t \) is related to the torsional angle \( \tau \) by the torsional constant \( C_t \):

\[
M_t = C_t \tau = \tau \int \left( C_{44} x^2 + C_{66} z^2 + C_{44} x \frac{\partial \phi}{\partial z} \right) dx \quad (7)
\]

\[
- C_{66} z \frac{\partial \phi}{\partial x} \right) dx \quad (7)
\]

Using (7), \( C_t \) is then expressed as:

\[
C_t = \frac{C_{66} x_o z_o^3}{3} \left( 1 - \frac{z_o \sqrt{C_{66}}}{C_{44}} \frac{192}{\pi^5} \right) \sum_{n=0}^{\infty} \left\{ \frac{(2n+1)^5}{(2n+1)^5} \frac{\pi^5}{\pi^5} \right\} \tanh \left( \frac{(2n+1)\pi x_o}{2z_o} \left( \frac{C_{44}}{C_{66}} \right) \right) \quad (8)
\]
The equilibrium equation (9) combined with stress equations (5.a & 5.b) lead to the equation of motion (10):

$$T_{y,j} = 0 \quad (9)$$

$$\rho I_y \frac{\partial^2 \alpha}{\partial t^2} - C_i \frac{\partial^2 \alpha}{\partial y^2} = 0 \quad (10)$$

where

$$I_y = \frac{x_o z_o (x_o^2 + z_o^2)}{12}$$

moment of inertia in rectangular cross-section

$C_i$: torsional constant

$\alpha$: rotation angle versus Y axis

The solution of the equation (10) is expressed as:

$$\alpha(y, t) = [C \sin(a y) + B \cos(b y)] \cos(\omega t) \quad (11)$$

The constants $C, B, a$ and $b$ are determined by the boundary conditions.

The resonant frequencies are determined by using the principle of energy’s conserving. Hence, for a beam which is fixed-free, the resonant frequencies are (12) and for a fixed-fixed beam they are expressed in (13):

$$f = \frac{n}{4} y_o \sqrt{\frac{C_i}{\rho I_y}} \quad (12)$$

$$f = \frac{n}{2} y_o \sqrt{\frac{C_i}{\rho I_y}} \quad (13)$$

In Table 1 are represented the resonant frequencies for the two boundary conditions expressed before and beam dimensions are: 1 mm of thickness, 2 mm of width and 15 mm of length:

<table>
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<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-free</td>
<td>40142</td>
<td>80285</td>
<td>120429</td>
<td>160572</td>
<td>200715</td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td>20071</td>
<td>40142</td>
<td>60214</td>
<td>80285</td>
<td>100357</td>
</tr>
</tbody>
</table>

Table 1.a : Quartz’s frequencies for the five first modes

### IV. Temperature effects of torsional modes

Using the approximate method, we obtain an analytical model of the first two TCF:

$$\alpha = \frac{1}{2} \left[ -\frac{y}{y} + \frac{C_{66}}{C_{66} + 3} \frac{\dot{z}}{z} + \frac{h}{1 - h} \frac{\ddot{h}}{h} \right]$$

$$-2 \frac{x^2}{x^2 + z^2} \frac{x}{x} - 2 \frac{\dot{z}}{x^2 + z^2} \frac{\ddot{z}}{z} \right]$$

$$\beta = \frac{1}{2} \left[ (\log(f))^2 + \alpha^2 \right] \quad (14.b)$$

with

$$\frac{\dot{h}}{h} = \frac{\frac{z}{x + z} + \frac{C_{66}}{C_{66} + 1} - C_{44}}{2 C_{66} \frac{1}{C_{44}}} (1\ldots)$$

$$A = \frac{(2n+1)\pi}{2z} \frac{x^2}{C_{66}} \left( \frac{C_{44}}{C_{66}} \right)^{\frac{1}{2}}$$

In Figure 4.a and 4.b are drawn the evaluation of the first TCF versus rotated X-cut for GaPO$_4$ and quartz.

![Figure 4: 1st TCF for quartz and GaPO$_4$ in torsional modes](image)

The temperature-compensated cut is found again for quartz at $\theta = -34^\circ$ ($\varphi = 60^\circ$) and $\theta = 32^\circ$ ($\varphi = 0^\circ$). GaPO$_4$ curve has a zero value for theta equal to $-7^\circ$ then it does exist a temperature-compensated cut for GaPO$_4$ in torsional modes.
In figure 5, we notice that the temperature effects have the same consequence on GaPO₄ and quartz for their temperature compensated cut. However, the excellent stability of GaPO₄ lets it to have a temperature compensated cut at high temperature. In figure 6, it is reported the frequency variations versus temperature for GaPO₄ for a cut (θ = 19.5°) which is compensated at 480 °C.

**Conclusion**

As a conclusion, we found again the cut angle where occurs the minimization of the temperature effects in flexural modes for quartz [2]. However for GaPO₄, a temperature compensated cut exists in flexural modes at about -16.7°, piezoelectric effects and mass loading need to be taking into account and its realize a little offset of the cut angle.

In torsional modes, quartz has got a temperature compensated cut at θ = -34° (φ = 60°) and θ = 32° (φ = 0°) [3]. GaPO₄ has also a temperature compensated cut at -7°.

For GaPO₄, by slightly adjusting the cut angle a temperature compensated cut exists at high temperature in torsional modes like quartz but material properties of quartz limit the temperature using at high temperature (<530 °C where α – β transition occurs). Whereas the excellent thermal stability of GaPO₄ lets using at high temperature (>530 °C). In flexural modes, for GaPO₄ temperature compensated cut exists at high temperature by slightly adjusting the cut angle too.

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**References**


