AN EFFICIENT ALGORITHM TO CALCULATE THE AQD DISTRIBUTION WITH EXPONENTIAL KERNEL FOR ULTRASONIC DOPPLER BLOOD FLOW MEASUREMENT

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Abstract

One of the main goals in ultrasonic Doppler blood flow measurement is the estimation of the mean speed. The aim of this work is focused in the Carotid artery. The Doppler signal's pseudo instantaneous mean frequency (*PIMF*) can be used to estimate the mean speed. Time frequency distributions (TFD) can be used to calculate the *PIMF*. The *PIMF* is the centroid of each TFD's time trace.

The problem consists on the huge amount of operations involved in the TFD calculation. In this work, an expression that calculates efficiently the discrete AQD time frequency distribution with a Gaussian exponential kernel is proposed. The required amount of operations is $O\left(N^2\right)$ for the simplified expression; while $O\left(N^3\right)$, for the original non-simplified one.

Introduction

Previously, the TFD Cohen class [1], such as Wigner Ville, Choi William, Bessel and Born Jordan distributions, has been used to calculate the signal's *PIMF*. Such studies have revealed that an efficient algorithm that calculates the Wigner Ville distribution, at $t=0$, is $O\left(N\log N\right)$. Also, efficient algorithms that calculate the other cited distributions are $O\left(N^2\right)$. This increment in the algorithmic complexity is compensated with a more accurate spectral estimation in the presence of noise. An alternative to further improve the precision of the spectral estimations, is the use of the adaptive Q-constant TFD class (AQD). The AQD class has advantages compared to the Cohen class since their kernels incorporate spectral information about the signal. Specifically, the AQD with Gaussian exponential kernel is investigated.

Continuous Adaptive Q-constant Distribution

The continuous adaptive Q-constant time-frequency distribution (AQD), with kernel $\phi\left(\theta,\tau\right)$, is defined by [2]:

$$S_x^{AQD}\left(t,f\right)=\frac{f}{f_r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^*\left(\frac{f}{f_r}\left(t-\mu\right),\frac{f}{f_r}\tau\right) \bullet x\left(\mu+\tfrac{1}{2}\tau\right)x^*\left(\mu-\tfrac{1}{2}\tau\right)e^{-j2\pi f\tau\frac{\lambda_x(\mu)}{\lambda_x(t)}}d\mu d\tau \quad (1)$$

where $\psi\left(t,\tau\right)$ is the Fourier transform of the kernel, f_r is a reference frequency and $\lambda_x\left(t\right)$ is the instantaneous frequency of the analysed analytic signal $x\left(t\right)$. Consequently, this distribution is signal adaptive. As usually, t and f are the time and frequency variables.

The kernel defines the specific properties about the time-frequency representation of the analysed signal.

Pseudo AQD Distribution

The pseudo AQD distribution is defined by:

$$PS_x^{AQD}\left(t,f\right)=\frac{f}{f_r}\int_{-T_W}^{T_W}W\left(\frac{\tau}{2}\right)W^*\left(-\frac{\tau}{2}\right)\int_{-\infty}^{\infty}\psi^*\left(\frac{f}{f_r}\left(t-\mu\right),\frac{f}{f_r}\tau\right) \bullet x\left(\mu+\tfrac{1}{2}\tau\right)x^*\left(\mu-\tfrac{1}{2}\tau\right)e^{-j2\pi f\tau\frac{\lambda_x(\mu)}{\lambda_x(t)}}d\mu d\tau \quad (2)$$

where $W\left(t\right)$ is a sampling window which satisfies the following conditions: $W\left(0\right)=1$; and, $W\left(t\right)=0$ for $\left|t\right|>\frac{1}{2}T_W$.

Discrete AQD Distribution

The discrete AQD distribution is a discrete version of (2). That is:

$$DS_x^{AQD}\left(n,k\right)=2\frac{k}{r}\sum_{\tau=-N+1}^{N-1}W\left(\tau\right)W^*\left(-\tau\right)\sum_{\mu=-M}^{M}\psi^*\left(-\frac{k\mu}{r},\frac{2k\tau}{r}\right) \bullet x\left(\mu+n+\tau\right)x^*\left(\mu+n-\tau\right)e^{-j\frac{2\pi k\tau}{N}\frac{\lambda_x(\mu+n)}{\lambda_x(n)}} \quad (3)$$

where $r=1$ is a reference frequency and $x\left(n\right)$ is a discrete analytic signal of length $L=2N-1$, whose elements are enumerated by $n=-N+1,\ldots,N-1$; the same for $W\left(n\right)$. As usually, n and $k=0,\ldots,N-1$ are the discrete time and frequency variables.

Gaussian exponential kernel

The Gaussian exponential kernel is defined by:

$$\phi\left(t,f\right)=\frac{1}{T\Omega}e^{-\frac{t^2}{2T^2}-\frac{(2\pi f)^2}{2\Omega^2}} \quad (4)$$

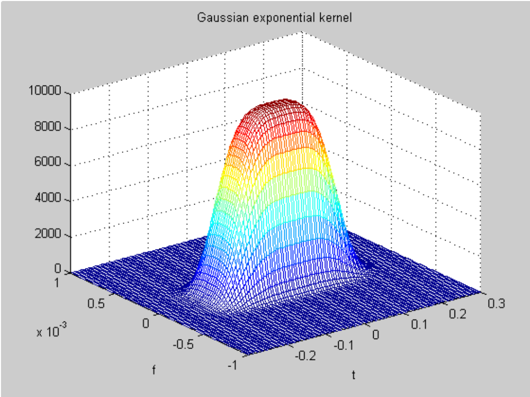

Figure 1 : Gaussian exponential kernel for $T = 0.1$ and $\Omega = 0.001$.

where $T > 0$ and $\Omega > 0$ are scaling parameters, whose product constitutes the Q factor. Their values produce a time-frequency smoothing in the analysed signal. Therefore, they establish the variance of the spectral estimations. A graphic of (4) is shown in Fig. 1.

Discrete AQD Distribution with Gaussian exponential kernel

Incorporating (4) in (3) conveniently and evaluating at $n = 0$, it is obtained:

$$DS_x^{AQD}(0,k) = \sqrt{\frac{2}{\pi}}\frac{k}{T}\sum_{\tau=-N+1}^{N-1} W(\tau)W^*(-\tau)e^{-2\Omega^2k^2\tau^2} \sum_{\mu=-N+1+|\tau|}^{N-1-|\tau|} e^{\frac{-k^2\mu^2}{2T^2}} x(\mu+\tau)x^*(\mu-\tau)e^{\frac{-j2\pi k\tau}{N}\left(\frac{\lambda_x(\mu)}{\lambda_x(0)}\right)} \quad (5)$$

The number of operations involved in (5) to evaluate the distribution for each k is $O(N^2)$. The whole calculation is $O(N^3)$. Also, it is no possible to use an FFT-like algorithm due to both the Gaussian exponential factors depending on k and the complex exponential factor depending on λ.

Algebraic simplification of the discrete AQD

The following algebraic simplification respect to τ is applied to (5) [3]. Firstly:

$$\sum_{\tau=-N+1}^{N-1} f(\tau) = \sum_{\tau=0}^{N-1} f(\tau) + \left[\sum_{\tau=0}^{N-1} f^*(-\tau)\right]^* - f(0)$$

Then, it is shown that:

$$f(\tau) = f^*(-\tau)$$

Finally:

$$\sum_{\tau=-N+1}^{N-1} f(\tau) = 2\text{Real}\left[\sum_{\tau=0}^{N-1} f(\tau)\right] - f(0)$$

where the $\text{Real}[\bullet]$ operator means the real part of a complex number. The resulting expression is:

$$DS_x^{AQD}(0,k) = \sqrt{\frac{8}{\pi}}\frac{k}{T}\text{Real}\left[\sum_{\tau=0}^{N-1} W(\tau)W^*(-\tau)e^{-2\Omega^2k^2\tau^2} \bullet \sum_{\mu=-N+1+|\tau|}^{N-1-|\tau|} e^{\frac{-k^2\mu^2}{2T^2}} x(\mu+\tau)x^*(\mu-\tau)e^{\frac{-j2\pi k\tau}{N}\left(\frac{\lambda_x(\mu)}{\lambda_x(0)}\right)}\right] - \sqrt{\frac{2}{\pi}}\frac{k}{T}\sum_{\mu=-N+1}^{N-1} e^{\frac{-k^2\mu^2}{2T^2}} x(\mu)x^*(\mu) \quad (6)$$

Although the number of operations involved in (6) to evaluate the distribution for each k is $O(N^2)$, the required amount of complex sums and products is just the half than that in (5). The whole calculation is also $O(N^3)$.

Optimum parameters T and Ω

The optimum parameters are those that minimise the RMS error in the spectral estimations [4]. The considered spectral estimations are the pseudo instantaneous mean frequency (*PIMF*) and the RMS mean bandwidth (MB_{RMS}). The analysed signal is a simulated ultrasonic Doppler one that models the Carotid artery blood flow mean speed.

In order to obtain the optimum parameters, a RMS error function for each one of the considered spectral estimations must be constructed. Those functions depend on both the parameter values and the sampling window length L. The explored domains are $0 < T < 1$ and $0 < \Omega < 1$. The procedure is the following.

Each one of the consecutive sampling windows of length L with maximum overlapping $(L-1)$ is analyzed. First, its discrete AQD evaluated at the center of the sampling window ($n = 0$) is calculated; second, its pseudo instantaneous power distribution (*PIPD*); and third, its *PIMF* and MB_{RMS}. Finally, the obtained values of the spectral estimations are compared with their theoretical values.

Figures 2 and 3 show the corresponding RMS error functions for the estimation of *PIMF* and MB_{RMS} respectively, with a sampling window length of $L = 63$.

The optimum parameters are $T = 0.1$ and $\Omega = 0.001$ in both cases.

Simulated ultrasonic Doppler signal

The simulated ultrasonic Doppler signal models the Carotid artery blood flow mean speed. It is known that the *PIMF* is proportional to the mean velocity. The theoretical *PIMF* is shown in figure 4. The theoretical MB_{RMS} is 100[Hz]. The sampling frequency is 25.5[kHz]. The detailed procedure to construct the signal can be found in [4][5].

Spectral estimations

The pseudo instantaneous mean frequency (*PIMF*) for an analytic signal is defined by:

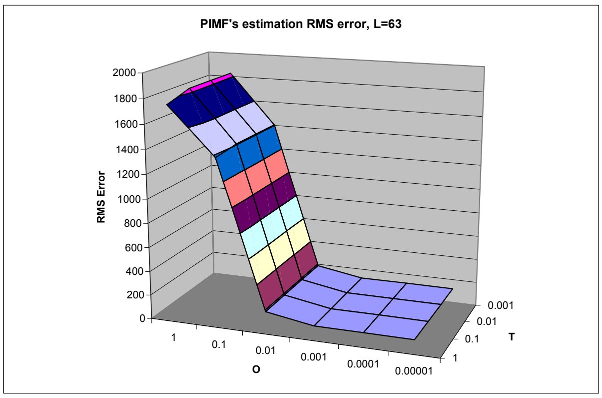

Figure 2 : RMS error function of the PIMF estimation with a sampling window length of $L = 63$.

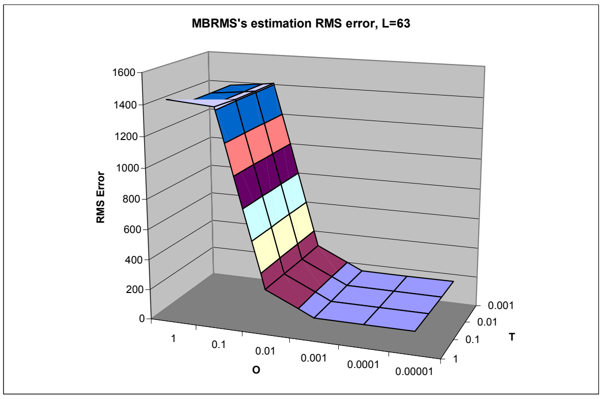

Figure 3 : RMS error function of the MB_{RMS} estimation with a sampling window length of $L = 63$.

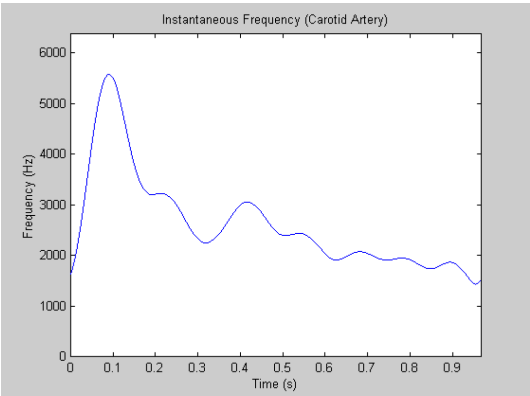

Figure 4 : Theoretical *PIMF* of the simulated ultrasonic Doppler signal.

$$PIMF(m) = \frac{\sum_{k=0}^{N/2-1} k \bullet PIPD(0,k)}{\sum_{k=0}^{N/2-1} PIPD(0,k)} \tag{7}$$

where the pseudo instantaneous power distribution (*PIPD*) is given by:

$$PIPD(0,k) = \begin{cases} DS_x^{AQD}(0,k) & DS_x^{AQD}(0,k) \geq 0 \\ 0 & DS_x^{AQD}(0,k) < 0 \end{cases} \tag{8}$$

On the other hand, the RMS mean bandwidth (MB_{RMS}) for an analytic signal is calculated by:

$$MB_{RMS}(m) = \sqrt{\frac{\sum_{k=0}^{N/2-1} \left(k - PIMF(m)\right)^2 \bullet PIPD(0,k)}{\sum_{k=0}^{N/2-1} PIPD(0,k)}} \tag{9}$$

Note that the m index corresponds to the m^{th} sampling window.

RMS error

For a set of estimated values, the error of each element is:

$$error_i = estimated_i - theoretical_i \tag{10}$$

and the RMS error of the whole set is:

$$error_{RMS} = \sqrt{\left(\overline{error_i}\right)^2 + \left(\sigma_{error_i}\right)^2} \tag{11}$$

Index optimization of the discrete AQD

The real exponential terms of the Gaussian kernel AQD decrease asintotically. Consequently, the elimination of the products that involve factors tending to zero in (6) is proposed. Specifically, if $\exp\left(-0.5k^2\mu^2/T^2\right)$ or $\exp\left(-2\Omega^2k^2\tau^2\right)$ are less than a threshold, then the corresponding values of the summation index μ or τ, respectively, are discarded.

The discrete AQD with optimized indexes is given by the following expression:

$$DS_x^{AQD}(0,k) = \sqrt{\frac{8}{\pi}}\frac{k}{T}\text{Real}\left[\sum_{\tau=0}^{\tau_{pot}(k)} W(\tau)W^*(-\tau)e^{-2\Omega^2k^2\tau^2} \bullet \sum_{\mu=-\mu_{pot}(k)}^{\mu_{pot}(k)} e^{\frac{-k^2\mu^2}{2T^2}} x(\mu+\tau)x^*(\mu-\tau)e^{\frac{-j2\pi k\tau}{N}\left(\frac{\lambda_x(\mu)}{\lambda_x(0)}\right)}\right] - \sqrt{\frac{2}{\pi}}\frac{k}{T}\sum_{\mu=-\mu_{pot}(k)}^{\mu_{pot}(k)} e^{\frac{-k^2\mu^2}{2T^2}} x(\mu)x^*(\mu) \tag{12}$$

where:

$$\tau_{opt}(k) = \max\left\{0 \leq \tau \leq N-1 \middle| e^{-2\Omega^2k^2\tau^2} \geq \text{tol}\right\} \tag{13}$$

$$\mu_{opt}(k) = \min\left\{N-1-\tau, \mu_{\max}(k)\right\} \tag{14}$$

$$\mu_{\max}(k) = \max\left\{0 \leq \mu \leq N-1 \middle| e^{\frac{-k^2\mu^2}{2T^2}} \geq \text{tol}\right\} \tag{15}$$

and tol is the threshold.

Figures 5 and 6 show the graphics of those real Gaussian exponential factors with the optimum parameters $T = 0.1$ and $\Omega = 0.001$.

Note that each one of the traces of $\exp\left(-0.5k^2\mu^2/T^2\right)$ for $k = 1,\ldots,N-1$ becomes a discrete impulse $\delta(\mu)$, when the exponential values less that the threshold are set to zero. The considered threshold is $tol = 0.00001$. Also, note that $DS_x^{AQD}(0,k=0) = 0$.

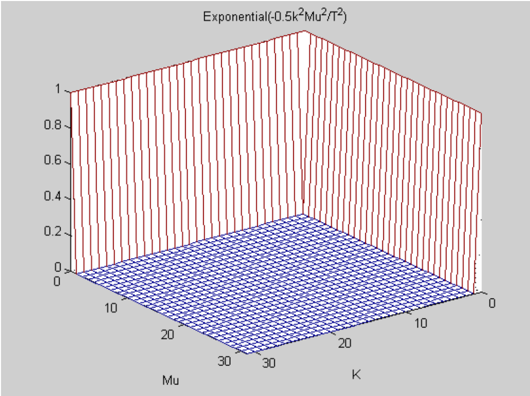

Figure 5 : Graphic of the real Gaussian exponential factor $\exp\left(-0.5k^2\mu^2/T^2\right)$ with $T=0.1$ and $L=63$.

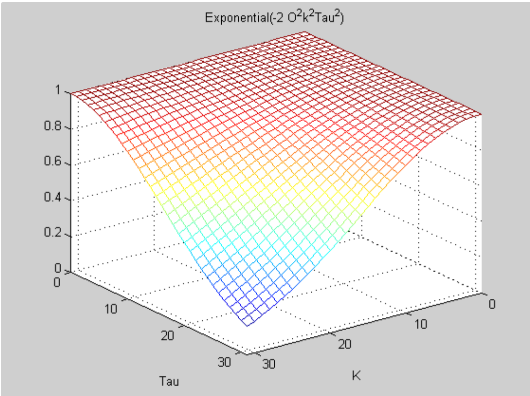

Figure 6 : Graphic of the real Gaussian exponential factor $\exp\left(-2\Omega^2k^2\tau^2\right)$ with $\Omega=0.001$ and $L=63$

Full simplification of the discrete AQD

As mention before, each one of the traces of $\exp\left(-0.5k^2\mu^2/T^2\right)$ for $k=1,\ldots,N-1$ and $T=0.1$ becomes a discrete impulse $\delta(\mu)$, when the exponential values less that the threshold ($tol=0.00001$) are set to zero. Consequently, if it is considered that:

$$e^{\frac{-k^2\mu^2}{2T^2}} \rightarrow \delta(\mu)$$

$$\sum_{\mu=-M}^{M} f(\mu)\delta(\mu)=f(0)$$

then the following full simplification of the discrete AQD (12) is proposed:

$$DS_x^{AQD}(0,k)=\sqrt{\frac{8}{\pi}}\frac{k}{\mathrm{T}}\mathrm{Real}\left[\sum_{\tau=0}^{\tau_{pot}(k)} W(\tau)W^*(-\tau)\right. \qquad (16)$$
$$\left.\bullet\, e^{-2\Omega^2k^2\tau^2}x(\tau)x^*(\tau)e^{\frac{-j2\pi k\tau}{N}}\right]-\sqrt{\frac{2}{\pi}}\frac{k}{\mathrm{T}}x(0)x^*(0)$$

with (13). The number of operations involved in (16) to evaluate the distribution for each k is $O(N)$. The whole calculation is $O(N^2)$. Note that (16) is a very closely approximation of (12), but only when $T\le 0.1$.

Conclusions

In this work, an expression that calculates efficiently the discrete AQD time frequency distribution with a Gaussian exponential kernel has been proposed. That is the expression (16). Note that its validity is restricted to T parameter values such that $T\le 0.1$.

The algorithm that estimates the whole discrete AQD by its definition, expression (5), is $O(N^3)$, i.e., the number of involved operations is proportional to N^3. While the algorithm that estimates completely the proposed simplified expression is just $O(N^2)$.

In order to achieve the efficient expression, two types of simplifications are used. The first one is an algebraic simplification (6). The second one takes into account that the real Gaussian exponential terms of the discrete AQD decrease asintotically. Consequently, the elimination of the products that involve factors tending to zero is proposed (12). Of course, this simplification considers the optimum parameters $T=0.1$ and $\Omega=0.001$ of the AQD. An optimum parameter minimizes the RMS error in the spectral estimations (*PIMF* and MB_{RMS}) and it depends on the analyzed signal. The last one is a simulated ultrasonic Doppler signal that models the Carotid artery blood flow mean velocity.

All the results are presented for a sampling window length of $L=63$, but they also applied to $L=127$ and $L=255$, indeed, in a more conveniently fashion.

Finally, the proposed expression (16) may be used to implement software applications that require a real-time response.

References

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