WAVE FORMS IN ADDITIONAL COMPONENTS OF ELASTIC BULK WAVES

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Abstract

Additional components in elastic wave field displacement are those which are zero for the case of propagation of a homogeneoeus-plane-wave. For *P*-waves in a homogeneous isotropic medium additional components are transverse components of the displacement.

We analyze simple exact non-nime-harmonic solutions of elastodynamics equations in order to find those which describe real wave fields more adequately. We find that theory of inhomogeneous plane waves and an alternative theory of elastic waves with a linear transverse structure give qualitatively different results.

Introduction

Interest in a detailed description of polarization in propagating elastic waves was motivated by possibilities of direct recording of three components of the displacement wave field. For example, in seismic exploration measurements in boreholes allow (after certain processing) complete description of the displacement vector. It was observed, i. e. [1], that the particle movement in *P*-waves is never rectilinear, and thus differs from predictions of a simple theory of homogeneous plane waves. We analyze simple solutions of elastodynamic equations in order to find those which describe real wave fields more adequately. We find that an alternative theory of propagation of elastic waves with a transverse structure give qualitatively different results.

The term "additional components" was introduced by people who developed an asymptotic ray approach, e. g. [2,3,4]. In case of isotropic elastic media, this meant, when considering e.g. *P*-waves its transverse components which necessarily appear in high-frequency perturbation theory when considering higher-order terms. The asymptotic theory, confirmed later by numerics, e.g. [5,6] predicted that the wave form in the additional component is the integral of the wave form in the component. We observe that this prediction fails for the usual inhomogeneous plane waves, e.g. [7]. We numerically simulate two kinds of non-trivial elastic wave fields which are both exact solutions of the elastodynamic equations. Both have non-zero additional components, but their wave forms are crucially different. In the first case it is the integral, and in the second it is the Hilbert transform of the wave form in the ``normal" component.

Mathematical background

We describe the physical elastic displacement vector in homogeneous isotropic medium by

$$\mathbf{U} = \Re(\mathbf{u}) \ ,$$

with $\Re$ standing for the real part, where $\mathbf{u} = \mathbf{u}(x, y, z, t)$ is a complex displacement vector obeying the standard elastodynamic equation

$$(l + 2m)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - m\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u} - r\frac{\partial^2 \mathbf{u}}{\partial t^2} = 0 \quad , \qquad (1)$$

with l and m the Lamé parameters and r the volume density, all constants.

Here we consider only *P*-waves, which can be represented in terms of the scalar potential $\Phi = \Phi(x, y, z, t)$ as follows

$$\mathbf{u} = \mathrm{grad}\ \Phi = \frac{\partial\ \Phi}{\partial x}\mathbf{e}_x + \frac{\partial\ \Phi}{\partial y}\mathbf{e}_y + \frac{\partial \Phi}{\partial z}\mathbf{e}_z\ . \qquad (2)$$

Here $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$ are unit coordinate vectors. Evidently (1) and (2) imply the wave equation for the potential

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0, \qquad (3)$$

where $c = \sqrt{(l + 2m)/r}$ is the velocity of bulk *P*-waves. We consider further potentials of three types, starting with the best-known one.

Model 1. Homogeneous plane wave

Taking a solution of (1) in the form of $\Phi = f(t - x/c)$ describing wave propagation in the direction of the x-axis, with f an arbitrary real function of a real variable, we get

$$\mathbf{U} = \frac{\partial f(t - x/c)}{\partial x}\mathbf{e}_x = F(t - x/c)\mathbf{e}_x \ , \qquad (4)$$

where

$$F(t) = -(1/c)\partial f(t)/\partial t \ . \qquad (5)$$

This solution describes rectilinear wave motion along $\mathbf{e}_x$, which is the``normal" component of the displacement, and no additional component is present.

Model 2. Homogeneous plane wave with a transverse structure

Under the same assumption about f, the potential

$$\Phi = (1 + Ay)\, f\,(t - x/c) \qquad (6)$$

where A is an arbitrary constant that we assume real. As immediately seen, the expression (6) satisfies (3), and thus the vector

$$\mathbf{U} = (1 + Ay)F(t - x/c)\mathbf{e}_x - Ac \int^{t-x/c} F(t)dt\, \mathbf{e}_y \qquad (7)$$

is an elastic wave displacement. The second term in (7) describes the anomalous or additional component, associated with the non-constancy of the wave field amplitude distribution along the wave front.

Expressions of the form (7) are typical for the ray theory involving higher-order terms [2,3,4,6].

Model 3. Inhomogeneous plane wave

Non-stationary plane-wave solution is described by the complex potential

$$\Phi = \frac{\tilde{c}}{c} f(t - \frac{x}{\tilde{c}} - i\frac{a}{\tilde{c}} y), \qquad (8)$$

where $0 < \tilde{c} < c$ is the velocity of the wave, $a = \sqrt{1 - \tilde{c}^2 / c^2}$ characterizes damping along the y-axis, and f is an arbitrary function of a complex variable.

The physical displacement is

$$\mathbf{U} = \Re(\mathrm{grad}\Phi) = \Re(F)\mathbf{e}_x + a\Re(iF)\mathbf{e}_y =$$

$$\Re(F)\mathbf{e}_x - a\Im(F)\mathbf{e}_y, \qquad (9)$$

with $\Im$ standing for the imaginary part and F still defined by (5) but the derivative with respect to the complex variable is meant. As known, real and imaginary parts of the function are connected under some natural assumptions, see, e. g. [8], by the Hilbert transform. Then

$$\mathbf{U} = \mathrm{F}(t)\mathbf{e}_x - a\ \mathrm{H[F]}(t)\mathbf{e}_y, \qquad (10)$$

where

$$\mathrm{F}(t) = F(t - \frac{x}{\tilde{c}} - i\frac{a}{\tilde{c}} y), \qquad (11)$$

and the Hilbert transform $\mathrm{H[F]}$ of the function F is defined as follows

$$\mathrm{H[F]}(t) = \frac{1}{p}\,\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{F(t)}{t - t} dt\ , \qquad (12)$$

with p.v. denoting the principal value of the integral.

In contrast to the previous case, the time-dependence of displacement in the additional component is not the integral but the Hilbert transform of that in the `normal' component.

Numerical simulation

Numerical modeling shows that wave forms in additional components described by models 2 and 3 with the same "normal" component may look very differently, see figures 1, 2 and 3.

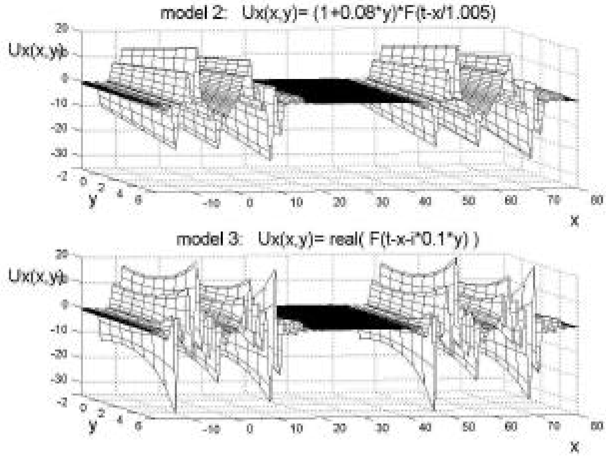

Figure 1: displacements for the direction of propagation (x-axis) for the models 2 and 3 and for two values of t

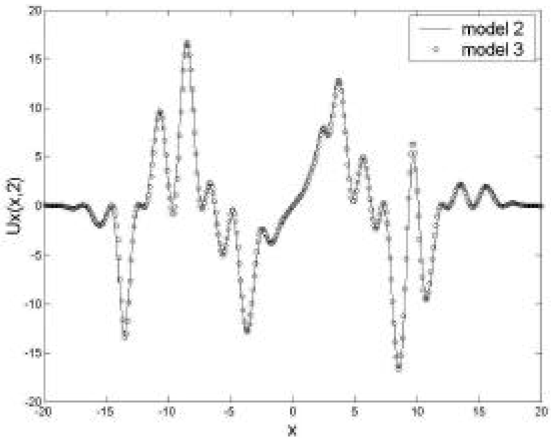

Figure 2: displacements for the direction of propagation for the models 2 and 3, y = 2, t=0

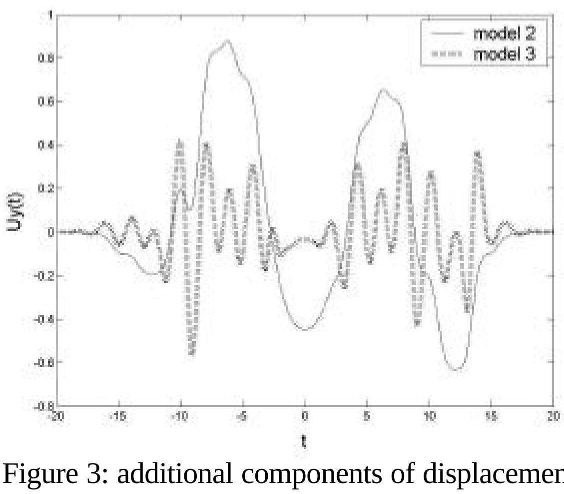

Figure 3: additional components of displacements for the models 2 and 3, x=0, y = 2

Conclusion

We observe that two natural models of the elastic *P*-wave field displacement demonstrate very different qualitative properties. At the moment we do not see any bridge between these models.

Similar results can be found for *S*-waves in isotropic media, and generalized to the case of general anisotropy.

The problem of adequateness of different mathematical models to physical realities remains open for both theoretical and experimental investigation.

Acknowledgement

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