A MODEL FOR THE NONLINEAR RESONANCE FREQUENCY SHIFT AND APPEARANCE OF HARMONICS IN DAMAGED MATERIALS

F. Windels[a] and K. Van Den Abeele[b]
[a]Kulak Nonlinear Acoustics Lab, IRC, KULAK, E. Sabbelaan 53, 8500 Kortrijk, Belgium
Post-doctoral researcher for the Fund of Scientific Research Flanders-Belgium, email: filip.windels@kulak.ac.be
[b]same adress as [a]

Abstract
A nonlinear version of Resonance Ultrasound Spectroscopy (RUS) theory is presented. This is important for NDT-purposes as damage manifests itself more pronounced and in an earlier stage by changes in the nonlinear elastical constants. General equations are derived for the 1-D case, describing the interaction between the modes due to the presence of a nonlinearity. An analytical solution of these equations is derived which predicts the shift of the resonance frequency versus amplitude and the appearance of harmonics in a bar with localized damage. This damage was modelled as a finite region, having a constant cubic nonlinearity, in an otherwise linear 1-D bar. The frequency shift and harmonic amplitudes can be experimentally determined by the SimonRUS technique. Finally the obtained formula was used to infer from the shifts in resonance frequency information about the position, and the non-linearity and width of the damage. Unlike other techniques, the proposed method does not require a scan to locate the defect, as it lets the different modes, having a different vibration pattern, probe the structure.

1. Introduction
RUS (Resonance Ultrasound Spectroscopy) [1] is a linear ultrasonic or acoustical technique where one extracts *all* the elastical constants of a sample out of its resonance frequencies, its geometry and its density. This method is very accurate when it is applied to samples having a well defined geometry and homogeneous elastical constants. It has been applied to determine the elastical constants of anisotropic media[2-4], to study thermoelectric materials [5], rocks [6],...

However, for NDT purposes, RUS has some drawbacks. As RUS is an inherent linear technique, it will not be so sensitive to the earlier stages of damage development inside a sample. This because, in contrast to intact materials, damaged materials exhibit not only a higher level of nonlinearity, but also the sensitivity of the variation of the nonlinearity with increasing damage is far better than what can be obtained from the evolution of the linear material parameters [7,8].

The above considerations underline the need for a nonlinear version of RUS: NRUS (Nonlinear RUS). This technique should work in a two-way direction. First, on the level of the direct problem, it should predict the nonlinear properties of the resonances from the nonlinear elastical constants inside the sample. A lot of experimental methods were already developed which exploit this principle: SimonRUS[8], NWMS[9],…. Complementary to this, numerical models were developed [10,11] which predict these effects. Nevertheless, these models lack the computational simplicity of linear RUS where the resonances can be directly determined as matrix eigenvalues[1]. So from the viewpoint of computational speed, physical insight and transparent formulas, a nonlinear variant of RUS would be highly desirable.
Moreover and secondly, on the level of the inverse problem, this would also be advantageous as how simpler one can solve the direct problem, how easier one can tackle the inverse problem. From this perspective an analytical solution for the direct problem would be the optimum.

Finally, the solution to the inverse problem would also be very useful to the problem of localization of nonlinear damage. Existing nonlinear damage localization techniques use the finiteness of either the acoustical source [12] or either the detector [13-15] to determine with a scan the damage position. For NRUS this would -in principle- not be required as the modes themselves do the scanning job: as the different modes have a different vibration pattern, they will probe different parts of the structure onto nonlinear properties. The detector and excitation source can therefore remain fixed (as in linear RUS), hence eliminating the need for a laborious scanning apparatus.

2. Derivation of the 1-D NRUS equations.

2.1. Equations in general coordinates
The 1-D nonlinear wave equation [16]

$$\rho\,\partial_{tt}^2 u = \partial_x\left\{K\,\partial_x u\left[1+\beta\,\partial_x u+\delta(\partial_x u)^2\right]\right\} \qquad (1)$$

corresponds to the Lagrangian:

$$L=\int_0^L dx\left\{\frac{\rho}{2}(\partial_t u)^2-\frac{K}{2}(\partial_x u)^2-\frac{K\beta}{3}(\partial_x u)^3-\frac{K\delta}{4}(\partial_x u)^4\right\}. \qquad (2)$$

This Lagrangian of Eq. (2) is a functional of the *u*-field, depending on two variables, and its first partial derivatives. Its corresponding action is:

$$S[u,\partial_x u,\partial_t u]=\int dt\, L[u,\partial_x u,\partial_t u]. \qquad (3)$$

To find the physical field, one decomposes the field into a sum over products [17]

$$u(x,t)=\sum_i \phi_i(x)\, q_i(t), \qquad (4)$$

where $\phi_i(x)$ are a set of chosen -and hence known- *spatial* functions, whereas the $q_i(t)$ are the new unknowns: a *set* of *temporal* functions.
By substituting Eq. (4) into Eq. (2), one gets a Lagrangian which is now a functional of a *set* of 1-D temporal functions and their first derivative: (unless otherwise stated, Einstein convention applies)

$$L[q_n,\partial_t q_n]=\frac{1}{2}M_{ij}\ \partial_t q_i \partial_t q_j-\frac{1}{2}K_{ij}\ q_i q_j -\frac{1}{3}\hat{B}_{ijk}\ q_i q_j q_k-\frac{1}{4}\hat{D}_{ijkl}\ q_i q_j q_k q_l \qquad (5)$$

where

$$M_{ij}=\int_0^L dx\ \rho\ \phi_i\phi_j \qquad (6a)$$

$$K_{ij}=\int_0^L dx\ K\ \partial_x\phi_i\partial_x\phi_j \qquad (6b)$$

$$\hat{B}_{ijk}=\int_0^L dx\ K\beta\ \partial_x\phi_i\partial_x\phi_j\partial_x\phi_k \qquad (6c)$$

$$\hat{D}_{ijkl}=\int_0^L dx\ K\delta\ \partial_x\phi_i\partial_x\phi_j\partial_x\phi_k\partial_x\phi_l \qquad (6d)$$

are further called the respective mass-, stiffness-, quadratic nonlinearity- and cubic nonlinearity-tensors. The physical field (and the corresponding physical set $q_i(t)$) can then be found from the principal of least action:

$$\delta S=0 \Leftrightarrow \frac{\partial L}{\partial q_n}-\frac{d}{dt}\left(\frac{\partial L}{\partial(\partial_t q_n)}\right)=0. \qquad (7)$$

After some calculations, Eq. (7) reduces to the following set of equations:

$$M_{nj}\partial_{tt}^2 q_j+K_{nj}q_j+\hat{B}_{njk}q_j q_k+\hat{D}_{njkl}q_j q_k q_l=0. \qquad (8)$$

Eq. (8) is seen to be a set of second order homogeneous differential equations which are nonlinear and coupled.

2.2. Equations in normal coordinates

By changing to normal coordinates $z_n(t)$ in stead of $q_n(t)$ [17], Eq. (4) becomes

$$u(x,t)=\sum_i \varphi_i(x)\, z_i(t) \qquad (9)$$

Eq. (8) can then be rewritten into a form where the mode-coupling occurs solely by the nonlinear interaction: (no sum over n)

$$\partial_{tt}^2 z_n+\omega_n^2 z_n+B_{njk}z_j z_k+D_{njkl}z_j z_k z_l=0, \qquad (10)$$

where the ω_n are the frequencies of the linear (low amplitude) resonances [1]:

$$\{\omega_n\}\text{solution of}\ \left|-\omega^2 M_{ij}-K_{ij}\right|=0 \qquad (11)$$

and the B and D tensors correspond to $\hat{B}$ and $\hat{D}$, but are now calculated for the normal coordinates. The procedure which leads to Eq. (10) is essentially choosing for the $\phi_i(x)$–functions in Eq. (4) the modal shapes $\varphi_i(x)$ of the linear resonances . This makes it easy to understand that the coupling at the linear level disappeared in Eq. (10) (linear modes are by definition uncoupled). However, as can be seen from Eqns. (6c) and (6d), modes are seen to couple to each other at the nonlinear level, when they all have a non-zero strain level at places where there is a nonlinearity present. Note finally that the z_n in Eq. (9) are mono-frequency signals in the linear case. Therefore the corresponding φ_i are called the (linear) mode shapes. In the nonlinear situation the z_n contain also other (and shifted) frequency components as harmonics arise.

3. Direct problem: 1-D bar with a localized cubic nonlinearity

Without loss of generality we consider for the rest of this paper only a cubic nonlinearity. Eq. (10) states in general that mode n can be generated due to the cubic nonlinear interaction of the modes j, k and l. If n, j, k and l are all equal to -let us say- m, it describes how the nonlinearity influences the m^{th}-mode, i.e. the m^{th}-resonance frequency will depend onto its amplitude. Therefore Eq. (10) should reveal the mechanism behind SimonRUS: how and why is the resonance frequency depending on its amplitude and the nonlinearity? The corresponding geometry is in Fig. 1. We take the normal coordinates for a 1-D bar ($\varphi_n(x)=\cos(n\pi x/L)$):

$$u(x,t)=\sum_n \cos\left(\frac{n\pi}{L}x\right) z_n(t) \qquad (11)$$

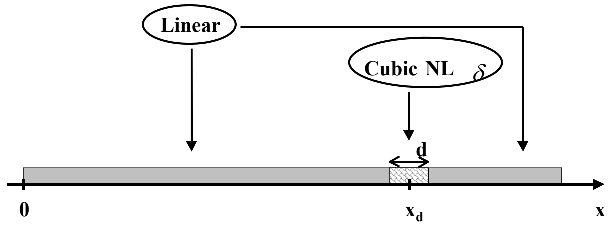

Fig.1: Geometry of the problem.

3.1 Shift of resonance frequency

For SimonRUS-applications we are interested in the influence of the nonlinearity onto the m^{th} resonance. The answer is given by Eq. (10) for $n=m$ and $j=k=l=m$ as the excitation is at mode m. (no sum over m.) Therefore only the following equation of Eqns. (10) applies in order to find the resonance shift:

$$\partial_{tt} z_m+\omega_m^2 z_m+D_{mmmm}z_m^3=0. \qquad (12)$$

where:

$$D_m = D_{mmmm} = \int_0^L dx\, K\delta \left(\partial_x \varphi_m\right)^4 . \qquad (13)$$

One can solve Eq. (12) by the method of multiple time scales[18,20].
As a result one finds that the resonance frequency is amplitude dependent, where *a* is the amplitude of the mode *m*, in the following way:

$$\omega'_m = \omega_m + \frac{3}{8\omega_m} D_m a^2 , \qquad (14)$$

It is seen that the resonance frequency ω_m has shifted to a value ω'_m, which agrees with the literature for nonlinear harmonic oscillators [19]. For a localized damage at $x=x_d$, extending from $[x_d - d/2, x_d + d/2]$ and having a constant cubic nonlinearity δ in this region like shown in Fig. 1, one can calculate the nonlinear coupling coefficient D_m to be (assume a constant integrand over the damaged zone)

$$D_m = K\delta d \left(\frac{m\pi}{L}\right)^4 \sin^4\left(\frac{m\pi}{L} x_d\right). \qquad (15)$$

Combining Eq. (15) and (14), leads to

$$\omega'_m = \omega_m + \frac{3}{4} v \frac{(m\pi)^3}{L^4} \sin^4\left(\frac{m\pi}{L} x_d\right) \delta d\, a^2 . \qquad (16)$$

The expression in terms of strain ε is more elegant:

$$\omega'_m = \omega_m \left[1 + \frac{3}{4} \sin^4\left(\frac{m\pi}{L} x_d\right) \delta \frac{d}{L} \varepsilon^2\right] \qquad (17)$$

Eq. (17) is an analytical formula which states that the resonance shift depends quadratically onto the amplitude, depends linearly onto the product of the nonlinear parameter δ and the relative length *d/L* of the defect, and depends like a $\sin^4$ onto the position of the damage x_d. This last dependence is quite understandable keeping Expressions (9), (11) and (13) in mind: only modes having a non-zero strain-field at the position of the defect will feel it. As a means of control of the $\sin^4$-dependence in Eq. (14), EFIT simulations [11] were done for the same problem. The values for the simulations can be found in Table 1. As illustrated in Fig. 2 the correspondence is striking and validates the NRUS predictions. On the vertical axis one sees the proportionality factor before the strain in Eq. (17).

Table 1

Parameter	Value
L	0.25 m
K	10^{10} Pa
ρ	2600 kg/m^3
V	1961 m/s
D	0.0125 m
δ	-10^6
Q	80
F_1	3922 Hz

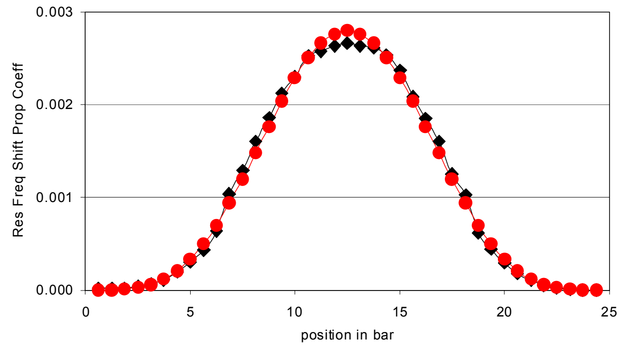

Fig.2: Sensitivity of the resonance frequency-shift to the damage location of the first mode. Circles (theory:$\sin^4$-fit of Eq.(17))-Diamonds (Simulation).

3.2 Amplitude 3rd harmonic

To find the 3rd harmonic amplitude, one chooses the following set of equations from Eqns. (10):

$$\partial_{tt} z_1 + 2\frac{\omega_1}{Q}\partial_t z_1 + \omega_1^2 z_1 + D_{1111} z_1^3 = F e^{i\Omega t} \qquad (18a)$$

$$\partial_{tt} z_3 + 2\frac{\omega_3}{Q}\partial_t z_3 + \omega_3^2 z_3 + D_{3111} z_1^3 = 0 \qquad (18b)$$

Note that we only have a cubic nonlinearity δ, and that we include damping and a forcing as this was supposed in the simulations. Note the indices. One can solve Eq. (18a) to obtain the lowest order solution for z_1 (which is the linear case), and then substitute this expression as a source term in Eq. (18b) for z_3. One can then solve Eq. (18b) up to a lowest order. One obtains as an expression for the strain of the third harmonic ε_3 at resonance:

$$\varepsilon_3 = Q \sin\left(\frac{3\pi}{L} x_d\right) \sin^3\left(\frac{\pi}{L} x_d\right) \delta \frac{d}{L} \varepsilon_1^3 , \qquad (19)$$

where ε_1 denotes the strain of the fundamental. On the vertical axis one sees the proportionality factor before the strain in Eq. (19).

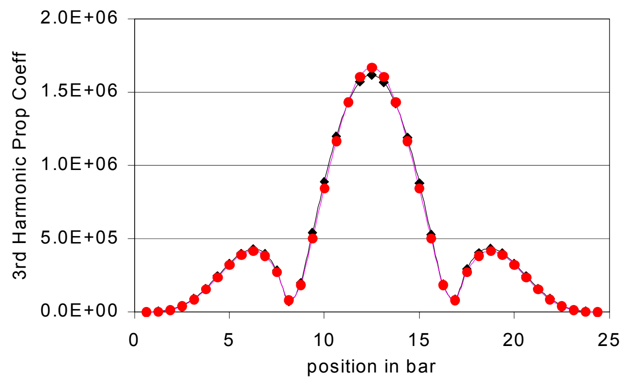

Fig.3: Sensitivity of the third harmonic amplitude to the location of the damage. Circles (theory: Fit according to Eq. (19))-Diamonds (Simulation).

4. Inverse problem

As the NRUS predictions of Eq. (17) are analytical in nature, they offer opportunities to solve the inverse problem: can one find the location of the defect, its nonlinearity and width out of the nonlinear behaviour of the resonances?

Consider the lowest two resonances, m=1 and m=2 (with respective strains ε_1 and ε_2). They both exhibit a resonance frequency shift given by (see Eq. (17)):

$$\frac{\Delta\omega_1}{\omega_1} = \frac{3}{4}\sin^4\left(\frac{\pi}{L}x_d\right)\ \delta\frac{d}{L}\varepsilon_1^2 \qquad (20a)$$

$$\frac{\Delta\omega_2}{\omega_2} = \frac{3}{4}\sin^4\left(\frac{2\pi}{L}x_d\right)\ \delta\frac{d}{L}\varepsilon_2^2 \qquad (20b)$$

By combining Eqns. (20a) and (20b) one can find x_d to be (ε_1=ε_2):

$$x_d / L = \frac{1}{\pi}\mathrm{Acos}\left[\sqrt[4]{\frac{\Delta\omega_2/\omega_2}{16\,\Delta\omega_1/\omega_1}}\right] \qquad (21)$$

and

$$\delta\frac{d}{L} = \frac{4\,\Delta\omega_1/\omega_1}{3\sin^4\left(\frac{\pi}{L}x_d\right)\varepsilon_1^2} \quad . \qquad (22)$$

Eq. (21) shows us that the position of the defect can be inferred from the frequency shift of two resonances, without knowing the absolute amplitude (i.e. no time-consuming calibration is required). These shifts can be experimentally determined by the SimonRUS-technique. It should be stated that the inversion-procedure by Eq. (21) is not unique as defects located symmetrically with respect to the center of the bar cannot be distinguished from each other. What concerns the nonlinearity, Eq. (22) states that it appears in combination with the width of the defect, hence $\delta d/L$ can be considered to be a kind of effective damage-parameter characterizing the defect zone.

5. Conclusions

A nonlinear version of Resonance Ultrasound Spectroscopy (RUS) theory was developed. General equations were developed in the 1-D case, describing the interaction between the modes due to the nonlinearity. These equations were solved up to a first order in the case of a cubic nonlinearity, which is constant in a limited region of a 1-D bar. This simulates a localized region of damage in an otherwise intact bar. The solution predicts that the nonlinear shift of the modal resonance frequency is quadratic with amplitude, proportional with the product of the nonlinearity and the width of the defect, and depends like $\sin^4$ onto the position of the defect. Also the dependency of the amplitude of the third harmonic onto the damage parameters was studied. Finally the obtained expressions were used to infer from the shifts in resonance frequency the position, and the nonlinearity and extension of the damage.

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