CALCULATION OF THE EXACT RESPONSE OF A TWO-LAYERED HALF-SPACE UNDER DYNAMIC POINT LOADING

Q. Grimal, S. Naili et A. Watzky
Laboratoire de Mécanique Physique - B2OA, CNRS UMR 7052
Université Paris XII-Val de Marne, Créteil, FRANCE
grimal@univ-paris12.fr

Abstract

This study presents a method to calculate the response of an elastic structure, made of a layer overlaying a half-space, loaded at its free surface by a dynamic normal point force. Both welded and frictionless sliding conditions at the interface are considered. The work is conducted in the context of characterizing the transmission of waves in the human lung (half-space) after non-penetrating impact on the thoracic wall (layer). Equations of the boundary-value problem are manipulated in a Laplace-Fourier transform domain and the Cagniard-de Hoop method for 3D problems is used to invert the transforms. The solution is free of approximation and has a form identical to that obtained by intuition with the generalized ray theory. Results illustrate the method. We analyse: 1) the influence of the contact condition on the propagation of body waves and head waves; 2) the response to impact loadings of durations up to 30 times the transit time of P-waves in the layer.

Introduction

In many cases, the Cagniard-de Hoop (CdH) method is the only method able to yield exact Green's functions for problems of transient wave propagation in elastic layered media. See [1, 2] for reviews of the method.

The work presented in this paper may be viewed as an illustration of the use of the CdH method in a simple layered medium made of a layer resting on a half-space.

The CdH method is associated with the generalized ray theory. A solution obtained within the framework of this theory consists in a sum of terms, called "generalized rays" (GR). Each GR represents the contribution of a specific wave to the response: body waves, surface waves and head waves, with longitudinal or transverse polarization. As a consequence, the analytic expression of the response at a given receiver includes more and more rays when the observation time increases.

In the general case, evaluation of each GR's contribution requires numerical calculations (because analytic solutions are not explicit); this is why numerical results with the GR theory have only been obtained for short time responses, that is, when only a few number of waves have arrived to a given receiver.

In the present study, two original aspects of the GR/CdH method are addressed: 1) for the analysis of the role of the interface condition (welded vs. frictionless sliding) in the response pattern; 2) for the calculation of long time responses, which necessitate the calculation of millions of rays. Some details on the adaptations of the standard method have been presented in references [3, 4, 5, 6].

Formulation of the problem

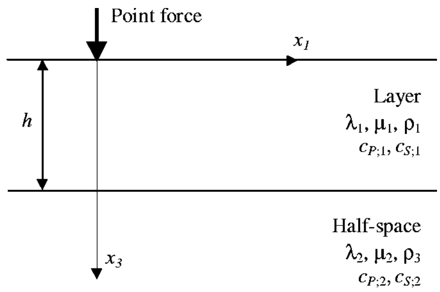

Figure 1: Model configuration and coordinate system.

The structure (Fig. 1) consists of a layer of infinite extent (medium 1) with thickness h overlaying a half-space (medium 2). The layer has a free surface (I), and an interface (II) with the half-space. At surface II, the contact condition is, alternatively, perfect bonding (welded contact) or frictionless sliding. Both media are linearly elastic, homogeneous and isotropic; the Lamé parameters are denoted by λ and μ, and the mass density is denoted by ρ (see Tab.1). Superscripts (1) and (2) refer to media 1 and 2, respectively (superscripts will be omitted in equations valid for both media).

Table 1: Material parameters and waves speeds in media 1 and 2.

λ_1 (MPa)	μ_1 (MPa)	ρ_1 (kg.m^{-3})	$c_{S;1}$ (m.s^{-1})	$c_{P;1}$ (m.s^{-1})
3 285	821	1 750	685	1 678
λ_2	μ_2	ρ_2	$c_{S;2}$	$c_{P;2}$
0.4	0.26	600	21	40

The position is specified through the coordinates (x_1, x_2, x_3) with respect to a Cartesian reference frame $\mathcal{R}(O;\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$ where O is the origin and $(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$ is an orthonormal basis for the space; the $\mathbf{x_3}$-axis is taken perpendicular to surfaces I and II. The free surface of the layer coincides with plane $x_3=0$ and the structure is localized in the half-space $x_3\geq 0$. Time is denoted by t. The elastic response is characterized in $\mathcal{R}$ by the components σ_{ij} of the Cauchy stress tensor and by the components v_i of the particle velocity $\mathbf{v}$.

Letters P and S are used for quantities relative to P- and S-waves. Wave speeds are defined by $c_P=\sqrt{(\lambda+2\mu)/\rho}$ and $c_S=\sqrt{\mu/\rho}$, and waves slownesses by $s_{P,S}=1/c_{P,S}$. At O, a dynamic point force of direction $\mathbf{x_3}$ generates both P- and S-waves with vertical polarization. Media are at rest for $t<0$.

With negligible body forces, the equation of motion is

$$\partial_j\sigma_{ij}-\rho\partial_t v_i=0,\quad i,j=1,2,3, \tag{1}$$

where ∂_j and ∂_t denote, respectively, partial derivatives with respect to x_j and to time, δ_{ij} is the Kronecker symbol and Einstein's summation convention is used. Hooke's constitutive law is introduced as

$$\partial_t\sigma_{ij}-\lambda\delta_{ij}\delta_{pq}\partial_q v_p-\mu(\partial_i v_j+\partial_j v_i)=0. \tag{2}$$

At surface I, the free surface conditions and the definition of loading are associated with the equations

$$\begin{array}{c}\sigma_{13}(x_1,x_2,0,t)=\sigma_{23}(x_1,x_2,0,t)=0\ ,\\ \sigma_{33}(x_1,x_2,0,t)=\sigma_0\phi(t)\delta(x_1)\delta(x_2),\end{array} \tag{3}$$

where $\phi(t)$ is the loading history, σ_0 is the loading strength ($\sigma_0=1$ in the computations) and δ is the Dirac function.

At surface II, frictionless sliding contact conditions are described by

$$\begin{array}{c}[\![v_3]\!]=0,\ [\![\sigma_{33}]\!]=0,\\ \sigma_{13}^{(1)}(x_1,x_2,h,t)=\sigma_{23}^{(1)}(x_1,x_2,h,t)=0,\\ \sigma_{13}^{(2)}(x_1,x_2,h,t)=\sigma_{23}^{(2)}(x_1,x_2,h,t)=0,\end{array} \tag{4}$$

where $[\![.]\!]$ denotes the jump of a quantity across the interface. And the welded contact conditions are

$$[\![\sigma_{13}]\!]=[\![\sigma_{23}]\!]=[\![\sigma_{33}]\!]=0;\quad [\![v_1]\!]=[\![v_2]\!]=[\![v_3]\!]=0. \tag{5}$$

Solution in the transform domain

The above equations are subjected to a Laplace transform with respect to t and a two-dimensional Fourier transform with respect to x_1 and x_2; p and k_i ($i=1,2$) are the Laplace and Fourier parameters, respectively. After elimination in the transform domain counterparts of (1) and (2) of stresses σ_{11}, σ_{12} and σ_{22}, six transform domain unknown quantities remain for representing the wave field in each medium; they are arranged into vector $\tilde{\mathbf{b}}=(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,-\tilde{\sigma}_{13},-\tilde{\sigma}_{23},-\tilde{\sigma}_{33})^T$. In both media, the differential equation for $\tilde{\mathbf{b}}$ takes the form [2]

$$\partial_3\tilde{\mathbf{b}}=-pA\tilde{\mathbf{b}}, \tag{6}$$

where A is a 6 by 6 matrix. Upon introducing $\bar{\mathbf{w}}$ as $\tilde{\mathbf{b}}=D\bar{\mathbf{w}}$, where each column of matrix D is an eigenvector of matrix A, (6) becomes

$$\partial_3\bar{\mathbf{w}}=-p\Lambda\bar{\mathbf{w}}, \tag{7}$$

where Λ is a diagonal matrix whose non-zero terms λ_i are the eigenvalues of A: $\lambda_1=-s_3^P$, $\lambda_2=-s_3^S$, $\lambda_3=-s_3^S$, $\lambda_4=s_3^P$, $\lambda_5=s_3^S$, $\lambda_6=s_3^S$, where $s_3^{P,S}=(s_{P,S}^2-k_1^2-k_2^2)^{1/2}$. In order to keep the square roots single valued in the derivations, $s_3^{P,S}$ are chosen so that $Re[s_3^{P,S}]\geq 0$.

The six solutions of (7) have the structure of inhomogeneous plane waves propagating in direction $\mathbf{x}_3$

$$\bar{w}_n=w_n\times\exp(-p\lambda_n x_3). \tag{8}$$

Solving the elastodynamics problem in the transform domain consists in determining $\tilde{\mathbf{b}}^{(1)}$ *via* $\bar{\mathbf{w}}^{(1)}$ in the layer and $\tilde{\mathbf{b}}^{(2)}$ *via* $\bar{\mathbf{w}}^{(2)}$ in the half-space; the interface conditions are used to couple $\bar{\mathbf{w}}^{(1)}$ and $\bar{\mathbf{w}}^{(2)}$. After some algebraic manipulations and by using an expansion in power series [7, 5],

$$\mathbf{w}^{(1)}=\sum_{k=0}^{\infty}R^k\mathbf{s}, \tag{9}$$

where $\mathbf{w}^{(1)}$ contains six amplitudes of waves in the transform domain. (See [7] for a discussion of the convergence of the sum in (9).) The components of $\mathbf{s}$ are amplitudes of the waves generated by the point force, and R is found to be a matrix which contains the reflection coefficients of plane waves at surfaces I and II. Eventually, upon developing (9), the solution is identified as a sum of generalized ray contributions. The counterpart of (9) in the space-time domain is a finite sum in a bounded observation time window. The solution $\bar{\mathbf{w}}^{(2)}$ in the half-space is obtained from the solution $\bar{\mathbf{w}}^{(1)}$ in the layer and has the same form.

Finally, in the half-space, the response is a sum of terms (each is the contribution of a single GR), denoted $[\tilde{b}_i^{(2)}]$: $\tilde{\mathbf{b}}^{(2)}=\sum_N[\tilde{b}_i^{(2)}]$, where N is the number of GRs to take into account in the response.

The form of one of the contribution is

$$[\tilde{b}_i^{(2)}] = D_{in}^{(2)}\ T_{II} \prod R_{lm}\ s_r\ \exp\left[-pg(s)\right], \quad (10)$$

where $g(s) = \sum s_3^{P,S;1} h + \lambda_n^{(2)}(x_3 - h)$, and the meaning of each quantity is indicated below.

$D_{in}^{(2)}$	polarization of the ray at the receiver;
T_{II}	transmission coefficient at surface II;
$\prod R_{lm}$	product of reflection coefficients at surfaces I and II;
s_r	amplitude coefficient at the source level;
p	Laplace parameter;
$g(S)$	"path" of the ray.

The expressions of $R_{I,II}$ and T_{II} depend on the contact condition considered at the interface.

Solutions in the time-space domain

Each GR contribution in the space-time domain is obtained by applying the Cagniard-de Hoop method. The principle of Cagniard method is to make one inverse Fourier transform and one Laplace transform play against each other so that they mutually annihilate themselves: the inverse Fourier transform of a GR contribution in the transform domain is manipulated; new contours of integration are defined. Together with the body wave associated with the GR under consideration, in some cases, an integration along branch cuts gives rise in the physical domain to head wave contributions and residues to interface waves. Exact solutions for the wave field are obtained; however they are rarely explicit for 3D problems.

While the above manipulations in the transform domain are indifferent to the location of the receiver, the form of the space-time domain solution strongly depends on this location. We do not give the technical detail in this paper, nor the expression of the solution, they can be found in references [2, 4, 6].

Results

The present work is part of a project which purpose is to characterize the transmission of energy in the lung following a non-penetrating impact on human thorax (lung is a biological tissue which has little resistance to dynamic loadings [8]).

The model shown in Fig. 1 is used to characterize the transmission of elastic waves between the thoracic wall (layer) and the lung (half-space). Material parameters are such that the layer is "hard" in comparison with the substrate—they are said to be "*weakly-coupled*". The thickness h of the layer is set to 2 cm.

A computer program based on the Cagniard-de Hoop method has been developed [5, 6]. The calculation of the response at a receiver in the half-space requires: *i*) to number all the rays arriving in a given time window and determine their expressions; *ii*) to evaluate numerically each ray contribution.

In the following, two kinds of numerical results are presented: 1) the response in the half-space due to the waves directly transmitted at the interface (those waves that have not undergone any reflection in the layer)—useful for the characterization of the role of the interface; 2) the response in the half-space, on the axis of symmetry, for various loading durations T—useful to elucidate the influence of the impact duration on the response.

Results 1) — Fig. 2 shows the response in terms of displacement u_1 at point $x_1 = x_3 = 0.025$ m in the half-space for the two different contact conditions and for $\phi(t) = H(t)$, where $H(t)$ is the Heaviside step function. Six waves arrive at the receiver: four body waves (PP, SP, PS and SS, in order of their arrival time) and two head waves (associated with GRs SP and SS). The bottom box in each figure shows the isolated contribution of head waves; the top figures are the total response (head waves+body waves).

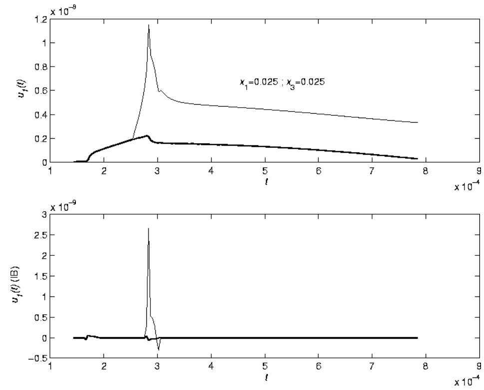

Figure 2: Response $u_1(t)$ at point $x_1 = x_3 = 0.025$ m for welded (thin line) and frictionless (thick line) contact; the head waves contributions are shown in the bottom figure.

It is seen on Fig. 2 that, $u_1(t)$ is very sensitive to the contact condition, and that the head wave contribution is very weak in the frictionless case as compared to the welded case. Other calculations have shown that ray SP and SS are much more influenced by the contact condition than PP and SP, and that this phenomenon is specific to weak coupling.

Results 2) — The response pattern of the structure is expected to be dependent on the duration T of the applied force with respect to the transit time $t_e = h \times s_P$ of P-waves to cross the layer thickness. The time history $\phi(t)$ is taken to be a four-point optimum Blackman window function [2] of unit amplitude. The response in the half-space is given in terms of σ_{33}, for both contact conditions, at a receiver placed on the $\mathbf{x}_3$-axis at 5 mm from the interface.

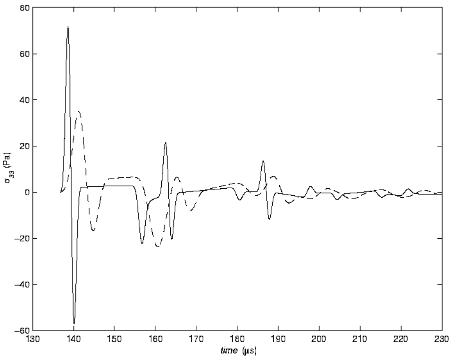

Figure 3: Stress $\sigma_{33}(t)$ for a receiver in the half-space at 5 mm from the interface for welded contact. (—) impulse duration T=5 μs; (- - -) T=11.919 μs (equal to the transit time in the layer).

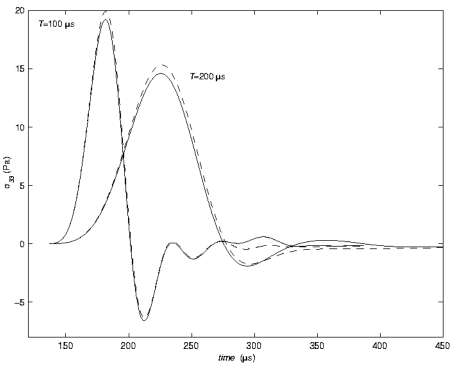

Figure 4: Stress $\sigma_{33}(t)$ at a receiver in the half-space at 5 mm from the interface for impulse durations T=100 μs and T=200 μs. (—) welded contact; (- - -) sliding contact.

Fig. 3 shows responses to "short" pulses, *i.e.*, of duration about or less than t_e; in these cases, responses obtained with sliding or welded contact were indistinguishable; the arrival times of the waves multiply reflected at the surfaces of the layer are manifest.

Fig. 4 shows responses to "long" pulses; in these cases, slight differences due to the contact conditions are observed; for loading durations of more than $10\times t_e$, the wave phenomena observed in Fig. 3 are replaced by vibrations of small amplitudes. For the longest loading duration, the response has almost the shape of the input pulse and vibrations disappears; the limit response for long duration loading matches the response of a plate (with the assumptions of the classical theory of plates).

Conclusions

Results presented in this paper illustrate the use of the exact 3D generalized ray/Cagniard-de Hoop method to investigate the transmission of transient waves within a substrate. The method proves to be a good candidate to test the influence of "ideal" contact conditions—welded contact or frictionless sliding. While the method is known to be efficient to obtain exact responses to short pulses, the results presented demonstrate that exact responses to long pulses can also be obtained (these calculations required the computation of millions of rays and were possible thanks to the low computational cost of the rays on the axis of symmetry). When it comes to modelling the response of structures in cases where nor high nor low frequencies approximations are valid, the fruits of the method may be worth the effort required to implement it, as an alternative to purely numerical methods.

ACKNOWLEDGMENT: The authors would like to thank the "Délégation Générale pour l'Armement" of the Minister of Defense of France for supporting this work.

References

[1] K. Aki and P. Richard, Quantitative Seismology: Theory and Methods. 1980.

[2] J. van der Hijden, Propagation of Transient Elastic Waves in Stratified Anisotropic Media, vol. **32**. 1987.

[3] Q. Grimal, S. Naïli, and A. Watzky, "A study of transient elastic wave propagation in a bimaterial modeling the thorax," International Journal of Solids and Structures, vol. **39**, pp. 5345–5369, 2002.

[4] Q. Grimal, S. Naïli, and A. Watzky, "On the transmission of transient elastodynamic waves at a sliding contact interface; application to a weakly coupled bimaterial," International Journal of Solids and Structures, in press, 2003.

[5] Q. Grimal, S. Naïli, and A. Watzky, "A method for calculating the axisymmetric response of a two-layered half-space under concentrated loading.," Journal of Sound and Vibrations, in press, 2003.

[6] Q. Grimal. PhD thesis, Université Paris XII, 2003. In preparation.

[7] C. Ma and G. Lee, "Transient elastic waves propagating in a multi-layered medium subjected to in-plane dynamic loadings. II. Numerical calculation and experimental measurement.," Proceedings of the Royal Society of London Series A, vol. **456**, pp. 1375–1396, 2000.

[8] Y. Fung, Biomechanics Motion, Flow, Stress, and Growth. 1990.