USING COHERENT WAVES TO EVALUATE DYNAMIC MATERIAL PROPERTIES

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Abstract

A method of estimating effective macroscopic material properties (stiffness and density) for heterogeneous elastic materials containing distributions of identical infinite cylindrical objects, randomly and uniformly distributed, is formulated based on the theory of elastic-wave multiple scattering. We show that the equivalent homogeneous medium possesses effective dynamic complex-valued density and stiffness, which are frequency-dependent. Dynamic explicit expressions, requiring minimal computational effort, are established in terms of both the wavenumber in the object-free host medium and the single-particle forward and backward scattering amplitudes in the far field. Predictions for the dynamic properties are presented in the cases of a SH wave propagating in a solid containing empty cavities and of an incident longitudinal wave interacting with a set of water-immersed solid bars. In both cases, the qualitative behavior of the dynamic properties is in agreement with physical expectations.

Introduction

Modeling of linear wave propagation in heterogeneous media, such as bubbly liquid [1] or even fiber-reinforced material composite [2, 3], is still a challenging problem in the investigation of internal microstructure. Derivations have been performed to predict both propagation of plane wave [1, 2, 4-7] and dynamic (frequency-dependent) effective material properties [3, 8]. The term *effective* is associated with the long wavelength assumption used to denote the macroscopic behavior of the continuous phase (host medium) combined with the discontinuous one (objects).

The aim of this work is to describe macroscopically the dynamic behavior of a two-phase medium in which the energy dissipation mechanisms are only due to the multiple scattering between the elastic objects contained in the lossless host medium. Independent explicit analytical expressions are derived for the effective frequency-dependent material properties (density and stiffness) of a two-phase medium, in terms of the scattering amplitudes for a single object in the host medium. Unlike previous works [1], no assumption about the expression of the effective density has been made. When the energy losses are induced only by multiple scattering, we show that the effective medium can be seen as a linearly dissipative material in which inertial coupling between the two phases cannot be neglected. In other words the effective stiffness and density are both frequency-dependent complex-valued functions.

After stating the problem, formulae for the dynamic material properties are established from the responses in transmission and reflection of a two-phase slab viewed by an incident plane wave as either heterogeneous or homogeneous. In this paper we refer to the notion of *heterogeneity* when we take into account the multiple scattering phenomena between the objects.

Problem statement

We consider a two-phase medium in which both phases are purely linearly elastic and isotropic. The two-dimensional objects are assumed to be identical, parallel, infinite and distributed randomly and uniformly. The incident wave travels normal to the symmetry axes of the objects. Material properties and acoustic field quantities of the host medium are indicated by the superscript 1 and those of the object by the superscript 0.

While the propagation of a plane monochromatic wave in the lossless object-free host medium is governed by the real-valued wavenumber k_1, it is well known [4, 6] that the coherent plane wave propagation in the two-phase medium can be conveniently described by an effective wavenumber, given by

$$K = \frac{\omega}{c} + \mathrm{i}\alpha \,, \tag{1}$$

where i is the imaginary unit, and ω is the angular frequency. In Eq. (1), c and α stand for the frequency-dependent effective phase velocity and effective attenuation. Occurred losses are here assumed to be induced only by multiple scattering, and anelastic attenuation is not taken into account. Note that no subscript is used for the properties of the equivalent homogeneous medium.

The problem to be solved is then reduced to expressing the material properties of the effective continuous medium from the effective wavenumber K. Assuming the effective continuous medium behaves as a linearly dissipative material, the dispersion relation can be written as [3]

$$\frac{\rho\,\omega^2}{K^2} = M\ , \tag{2}$$

where ρ is the effective density and M is the effective stiffness which relates linearly the spectra of strain and stress. The main effect of the dissipative processes, in a macroscopic description of sound propagation, is that in the dispersion relation (2), the density and the stiffness must be complex-valued quantities. While the density $\rho = \rho' + \mathrm{i}\,\rho''$ expresses the geometry-dependent inertial coupling between the individual materials or phases, the stiffness $M = M' - \mathrm{i}\,M''$ represents the constitutive law of the continuous medium.

Determination of the dynamic material properties

The dispersion relation (2) by itself is not sufficient to extract both the effective density and the effective stiffness from the effective wavenumber K. In the following, the necessary additional relation is obtained from the comparison between the wave amplitude transmission T and reflection R responses of a two-phase slab, viewed as either heterogeneous or homogeneous. The thickness of the slab is denoted d.

Multiple scattering approach

From the analysis of the coherent displacement field in (and out of) the slab, we find that the wavenumber K takes the following form [7]

$$\frac{K^2}{k_1^{\,2}} = \left(1 + \frac{2\pi\,n_0}{k_1^{\,2}}\,f(0)\right)^2 - \left(\frac{2\pi\,n_0}{k_1^{\,2}}\,f(\pi)\right)^2 \tag{3}$$

and that the coherent responses in transmission and reflection of the two-phase slab to normally incident SH or longitudinal waves are the following over the entire frequency range [7]

$$T = \frac{1-Q^2}{1-Q^2\,e^{2\mathrm{i}Kd}}\,e^{\mathrm{i}\,(K-k_1)\,d} \tag{4}$$

and
$$R = -\frac{Q\left(1-e^{2\mathrm{i}Kd}\right)}{1-Q^2\,e^{2\mathrm{i}Kd}}\,e^{-\mathrm{i}\,k_1\,d}\ , \tag{5}$$

with
$$Q = \frac{A_-\left(K-k_1\right)}{A_+\left(K+k_1\right)}, \tag{6}$$

where
$$A_\pm = b_+ K \pm b_-\left(k_1 - 2\mathrm{i}b_+\right)$$

and
$$b_\pm = \frac{\mathrm{i}n_0\pi}{k_1}\left(f(0) \pm f(\pi)\right).$$

The dependence of Eq. (3) in terms of the object density n_0 and the single-object backward $f(0)$ and forward $f(\pi)$ scattering amplitudes in the far field is identical to the one established by Waterman and Truell [4] for point-like objects. Finally, by combining Eqs. (3), (4) and (5), the coefficients T and R can be explicitly rewritten as functions of the scattering amplitude f, the object density n_0, the wavenumber k_1 in the host medium and the slab thickness d.

Homogeneous approach

In this subsection, we focus on the normal-incidence transmission and reflection coefficients for a homogenous slab with a linearly dissipative response embedded in a purely elastic material. Considering the continuity of the normal displacement and stress across both interfaces, we obtain the same formulae as those given in Eqs. (4) and (5) in term of the Q parameter dependence. The Q parameter (6) depends in this case on the acoustic impedances of both media, z_1 and Z, and is given by

$$Q = \frac{z_1 - Z}{z_1 + Z}. \tag{7}$$

In addition, observe that the cases of the SH and longitudinal waves differ by a minus sign in the expression (5) of the wave amplitude reflection coefficient R. Therefore, in contrast to the multiple scattering approach, the coefficient R, Eq. (5), depends on the nature of the incident plane wave.

Finally writing the acoustic impedances as functions of the material density and of the wavenumber,

$$z_1 = \frac{\rho_1\omega}{k_1} \text{ and } Z = \frac{\rho\omega}{K}, \tag{8}$$

allows one to express the coefficients T and R in terms of the material densities, ρ_1 and ρ, and the wavenumbers, k_1 and K.

Dynamic material properties

Comparison of the analytical expressions for T and R established in the two previous subsections leads us without difficulty to a formula for the effective density ρ in terms of the scattering amplitude, the wavenumber k_1 and the object density n_0. By using Eqs. (2) and (3), the dimensionless effective density $\tilde{\tau}$ and stiffness $\tilde{M}$, defined by

$$\tilde{\tau} = \tilde{\tau}' + \mathrm{i}\,\tilde{\tau}'' = \frac{\rho}{\rho_1} \text{ and } \tilde{M} = \tilde{M}' - \mathrm{i}\,\tilde{M}'' = \frac{M}{\rho_1\,c_1^{\,2}}\ , \tag{9}$$

are then functions of frequency. Normalization has been performed with respect to the lossless host-medium properties, where c_1 represents the phase velocity of the incident wave. We find then for incident SH waves

$$\tilde{\tau} = 1 + \frac{2\pi\,n_0}{k_1^{\,2}}\left(f(0) + f(\pi)\right) \tag{10}$$

and $$\tilde{M} = \left(1 + \frac{2\pi\, n_0}{k_1^{\,2}}\big(f(0) - f(\pi)\big)\right)^{-1}, \qquad (11)$$

while for incident longitudinal waves

$$\tilde{\tau} = 1 + \frac{2\pi\, n_0}{k_1^{\,2}}\big(f(0) - f(\pi)\big) \qquad (12)$$

and $$\tilde{M} = \left(1 + \frac{2\pi\, n_0}{k_1^{\,2}}\big(f(0) + f(\pi)\big)\right)^{-1}. \qquad (13)$$

Numerical simulation

The suitability of the formulae (10)-(13) for the effective dynamic material properties of two-phase media is examined in two examples. The cases of incident SH (longitudinal) waves propagating in a solid (non-viscous liquid) containing a random and uniform collection of cylindrical empty cavities (solid bars) are examined in the following.

Due to the nature of both the incident plane wave and the constituents composing the studied two-phase media, the effective stiffness $\tilde{M}$, Eq. (2), represents either the shear stiffness for the present case of SH incident plane waves or the compressive stiffness for the case of longitudinal incident plane waves. The volume fraction ϕ of cylindrical objects of radius a is defined by $\phi = n_0\, a^2 \pi$.

SH waves in a solid with cylindrical empty cavities

Fig. 1 shows the frequency-dependent dimensionless effective density $\tilde{\tau}$, Eq. (10), and shear stiffness $\tilde{M}$, Eq. (11), over a frequency range related to the homogeneous assumption. In all the figures, where we choose the dimensionless frequency lower than 1, the quantity $\lambda/(2\,a)$ is greater than π, λ representing the wavelength of the incident wave in the host medium.

As seen in Fig. 1, the effective material properties, $\tilde{\tau}$ and $\tilde{M}$, are complex-valued quantities and their frequency dependence is not negligible. The real parts of $\tilde{\tau}$ and $\tilde{M}$ are less than 1. The empty cavities affect the initial material properties of the host medium by making them lower. The two-phase medium is therefore less dense and less stiff than the host medium. This shows weakening of the host medium by the presence of the empty cavities.

When the object density tends to 0, the real parts of $\tilde{\tau}$ and $\tilde{M}$ tend to 1, while both imaginary parts tend to 0. This points out that the deviation between the properties of the host and two-phase media decreases with the volume fraction of empty cavities.

From a physical point of view, all these results are qualitatively acceptable.

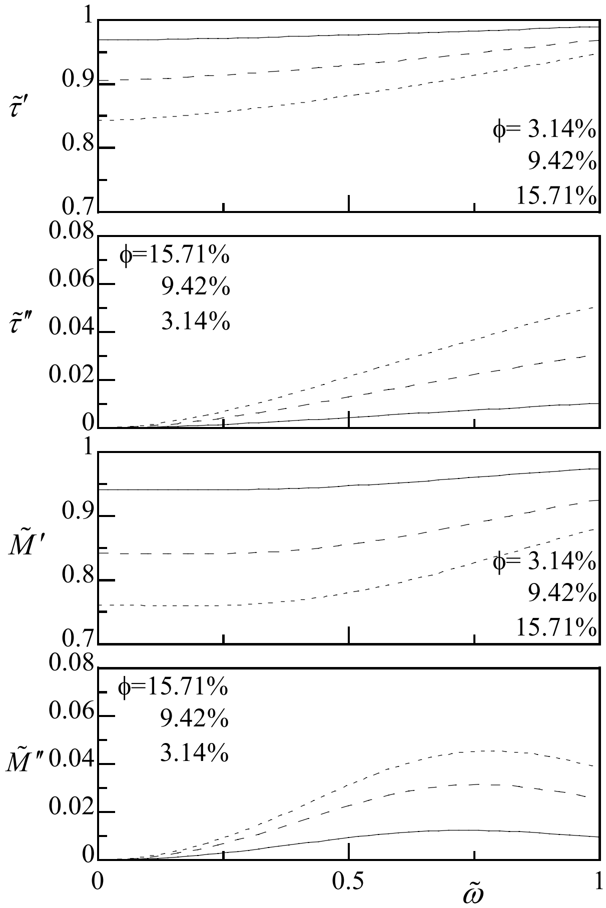

Figure 1: Dynamic material properties of a solid medium containing cylindrical cavities and subjected to incident SH waves: the density $\tilde{\tau}$ and the shear stiffness $\tilde{M}$ versus the frequency $\tilde{\omega}$ varying the volume fraction ϕ.

Examination of the effective material properties, Eqs. (10) and (11), in the long wavelength limit shows that the dimensionless effective density and shear stiffness become as follows in the static limit

$$\tilde{\tau}(\tilde{\omega} = 0) = 1 - \phi, \qquad (14)$$

$$\tilde{M}(\tilde{\omega} = 0) = \frac{1}{1 + 2\,\phi}. \qquad (15)$$

While Eq. (14) is identical to the one defined by the mixture rule, the static value of $\tilde{M}$ matches well, in the dilute case $(\phi \ll 1)$, with already-established expressions [2, 3].

Longitudinal waves in a non-viscous fluid with solid bars

Results for the prediction of the effective material properties of the two-phase medium subjected to a compressional perturbation, Eqs. (12) and (13), are displayed in Fig. 2, for different values of the bar concentration, as a function of the dimensionless frequency $\tilde{\omega}$. The dimensionless parameters representative of both phases have been chosen as

$$\tilde{\rho} = \frac{\rho_0}{\rho_1} = 9\,,\ \tilde{\kappa} = \frac{c_0^s}{c_0^\ell} = 0.5 \text{ and } \tilde{\kappa}_1 = \frac{c_1}{c_0^\ell} = 0.3 \quad (16)$$

to correspond approximately to the case of steel bars immersed in water. In Eq. (16), ρ_0, c_0^ℓ and c_0^s stand for the density, the phase velocity of the longitudinal and shear waves in the bars, respectively.

In contrast to the SH case, the bars affect the material properties of the non-viscous fluid by increasing the effective density and the compressive stiffness of the two-phase medium. This shows strengthening of the host medium by the presence of bars. All other comments made in the previous subsection concerning Fig. 1 remain valid.

Finally, the values in the low-frequency limit of the dimensionless effective density and compressive stiffness

$$\tilde{\tau}\left(\tilde{\omega}=0\right) = 1 + 2\,\phi\left(\frac{\tilde{\rho}-1}{\tilde{\rho}+1}\right), \quad (17)$$

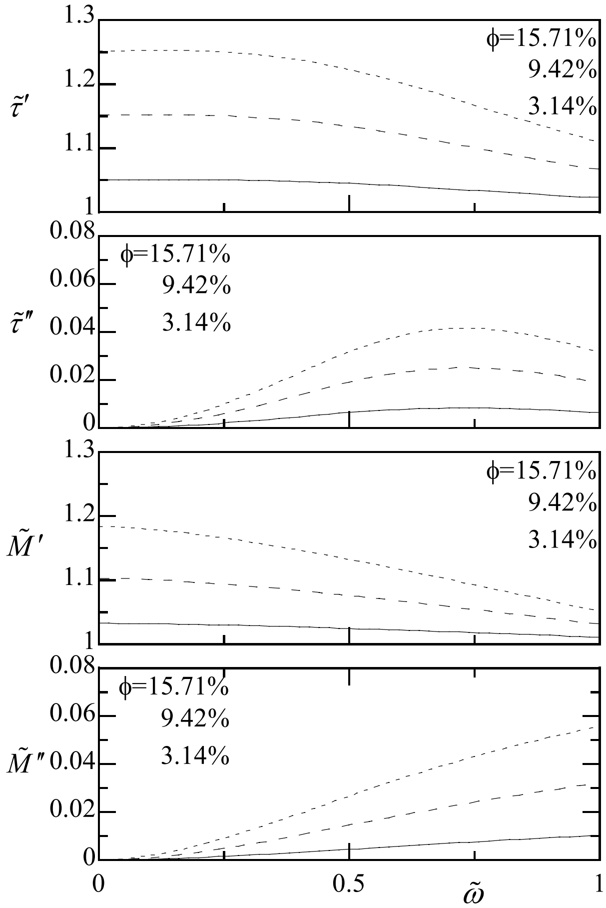

Figure 2: Dynamic material properties of water-immersed solid bars subjected to incident longitudinal waves: the density $\tilde{\tau}$ and the compressive stiffness $\tilde{M}$ versus the frequency $\tilde{\omega}$ varying the volume fraction ϕ.

$$\tilde{M}\left(\tilde{\omega}=0\right) = 1 - \frac{\phi\left(\tilde{\kappa}_1^{\,2} - \tilde{\rho}\left(1-\tilde{\kappa}^2\right)\right)}{\tilde{\rho}\left(1-\tilde{\kappa}^2\right) + \phi\left(\tilde{\kappa}_1^{\,2} - \tilde{\rho}\left(1-\tilde{\kappa}^2\right)\right)} \quad (18)$$

show that the static behavior of the considered two-phase medium is purely elastic. The difference between the effective density given in Eq. (17) and that satisfying the mixture law, $\tilde{\tau} = 1 + \phi\left(\tilde{\rho}-1\right)$, has already been explained by the relative motion between the bars and fluid.

Conclusion

Independent analytical formulae for the effective frequency-dependent density and stiffness of two-phase media are derived for a lossless host medium subjected to a SH or compressional disturbance and containing two-dimensional lossless objects distributed uniformly and randomly. We demonstrate that two-phase media, in which energy dissipation mechanisms are assumed to be only due to multiple scattering phenomena, have a linearly viscoelastic response in which the inertial coupling between the phases cannot be ignored.

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