CALCULATION OF THE VELOCITY SPECTRUM OF THE VERTICALLY INHOMOGENEOUS PLATES

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Abstract
The Peano expansion of the matricant provides a convenient framework to calculate the plane-wave dispersion spectra in elastic plates, continuously inhomogeneous through the plate. This formulation of the matricant is the analytical exact solution of the wave equation for such media. A commonly used alternative calculation is based on modelling the continuously inhomogeneous plate by a stack of homogeneous layers (the Thomson-Haskell method and its modifications) [1, 2, 3]. With the Peano expansion of the matricant, the solution may be built while retaining the continuity of the profiles. We present the dispersion curves obtained by the Peano expansion method for plates with different types of continuous inhomogeneity profiles of elastic properties and highlight clustering features of the spectra (acoustic channels).

Introduction
Continuous variation of material properties is a broadly encountered feature. Lateral inhomogeneity may occur naturally (for example the sea or the earth have depth-dependent properties), or else is due to damage (chemically attacked concrete) or manufacturing techniques (gluing, welding or residual stress) which create continuous transition zones. Most often the approach used to calculate the velocity spectrum of inhomogeneous structures relies on modelling the continuous profiles of material properties by piecewise constant functions. This leads to seeking an exact solution of an actually modified problem and adds some questions of accuracy and validity of the results in computing. In this paper we deal with an authentic problem keeping the continuity of the properties variation, and use its exact solution in the form of a matricant calculated by the way of the Peano expansion. Our goal is to show the feasibility of this method in computing velocity spectrum of a free inhomogeneous plate, by presenting two types of continuous inhomogeneity profiles, either gradually varying or including several acoustic channels.

Governing equations
Consider propagation of monochromatic elastic waves in a laterally inhomogeneous plate of thickness d. The displacement **U** and the traction vector **t** are sought in the form :

$$\mathbf{U} = \mathbf{u}(x_3)\exp[i(\omega t - k_1 x_1)],$$
$$\mathbf{t} = \mathbf{x}_3.\boldsymbol{\sigma}(x_3)\exp[i(\omega t - k_1 x_1)], \qquad (1)$$

k_1 being the horizontal wave number, ω the angular frequency and $\mathbf{x}_3$ the unit vector orthogonal to the plate faces (see Fig. 1).

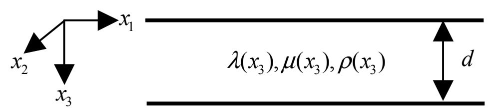

Figure 1 : Laterally inhomogeneous isotropic plate. λ and μ are the Lamé coefficients and ρ is the mass density.

The wave equation for displacement field is a second order differential equation with varying coefficients for which no explicit analytical solution generally exists unless appropriate profiles are involved allowing some special functions representation (Hankel, Whittaker) of the wave field [4, 5, 6].
The state vector approach in a space-time Fourier domain leads to the matrix differential equation of the first order [7] :

$$\frac{\mathrm{d}\,\mathbf{b}(x_3)}{\mathrm{d}\,x_3} = \mathbf{A}(x_3)\mathbf{b}(x_3)\,, \qquad (2)$$

where $\mathbf{b} = (u_i, \sigma_{i3})^T$ is the state vector (displacement – traction) and **A** is the system matrix depending on the elastic properties of the inhomogeneous medium, and on k_1 and ω.
In the case of an isotropic plate, the matrix equation (2) splits into two equations describing :

- SH- waves, x_2-polarized;
- P-SV coupled waves, (x_1, x_3)-polarized.

For SH-waves, the state vector and the system matrix are in this form :

$$\mathbf{b}(x_3) = \begin{pmatrix} u_2 \\ \sigma_{23} \end{pmatrix}$$
$$\mathbf{A}(x_3) = \begin{pmatrix} 0 & \mu(x_3)^{-1} \\ k_1^2\mu(x_3) - \omega^2\rho(x_3) & 0 \end{pmatrix}, \qquad (3)$$

and for P-SV waves :

$$\mathbf{b}(x_3) = \begin{pmatrix} u_1 \\ u_3 \\ \sigma_{13} \\ \sigma_{33} \end{pmatrix}$$

$$\mathbf{A}(x_3) =$$

$$\begin{pmatrix} 0 & ik_1 & \mu(x_3)^{-1} & 0 \\ \dfrac{ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)} & 0 & 0 & \dfrac{1}{\lambda(x_3)+2\mu(x_3)} \\ -\rho(x_3)\omega^2 + k_1^2\xi(x_3) & 0 & 0 & \dfrac{ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)} \\ 0 & -\rho(x_3)\omega^2 & ik_1 & 0 \end{pmatrix}$$

$$\xi(x_3) = \frac{4\mu(x_3)(\lambda(x_3)+\mu(x_3))}{\lambda(x_3)+2\mu(x_3)}. \qquad (4)$$

Solutions of the wave equation

The matricant and the Peano expansion [8]

The matricant $\mathbf{M}(x_3,\ x_3^{(0)})$ is the fundamental solution of Eq. (2) with the property :

$$\mathbf{b}(x_3) = \mathbf{M}(x_3,\ x_3^{(0)})\mathbf{b}(x_3^{(0)})\ , \qquad (5)$$

where $x_3^{(0)}$ is a reference point on the x_3-axis.
The matricant can be calculated by means of the Peano expansion :

$$\mathbf{M}(x_3, x_3^{(0)}) = \mathbf{I} + \int_{x_3^{(0)}}^{x_3} \mathbf{A}(\xi_1)d\xi_1 + \int_{x_3^{(0)}}^{x_3} \mathbf{A}(\xi_1) \int_{x_3^{(0)}}^{\xi_1} \mathbf{A}(\xi_2)d\xi_2 d\xi_1 + \ldots \qquad (6)$$

Matricant polynomial form

The explicit form of the system matrix **A** allows a convenient factorization by ω. In Eq.(3), **A** can be rewritten as a series in ω with coefficients depending only on x_3 (through the upper integration limit) and s_1, the horizontal component of the slowness. This factorization leads to a matrix polynomial form of the matricant. The coefficients of this polynomial are obtained for a fixed value of s_1 and are independent of the frequency.

Boundary problem

Let the plate faces be subjected to the traction-free boundary condition :

$$\sigma_{i3}(0) = \sigma_{i3}(d) = 0 \ \ (i = 1,2,3). \qquad (7)$$

Combining (8) with (5) leads to the dispersion equation in the form :

$$\det \mathbf{M_3}(d,0) = 0, \qquad (8)$$

where $\mathbf{M}_3$ is the left off-diagonal block of **M**. The left-hand side of Eq (8) can be arranged as a polynomial (see above), whose zeros are the eigen frequencies of the structure for a prescribed value of the horizontal slowness s_1. Thus obtained, the set of pairs (ω, s_1) describe the dispersion curves. Finding for each (ω, s_1) the null vector $\mathbf{u}(0)$ of $\mathbf{M}_3(d,\ 0)$ and using Eq (5) enable us to deduce the displacement - traction field at any x_3.

Accuracy and validity

In the Peano expansion method the degree of accuracy relies firstly on the numerical evaluation of the integrals and secondly on the truncation of the series. The order of accuracy of numerical integration methods is well known. So, any desirable accuracy can be reached by choosing an appropriate way of calculation.

Results

For these calculations, 100 terms are retained in the series, which is sufficient by a wide margin to ensure the convergence of the series for the range of frequency considered.

Linear profile of velocities

A linear profile is considered as a simple example. The plate properties vary linearly from titanium constants to Alpha Case (phase of titanium appearing in defective joints) properties. The properties of those materials are, for titanium, C_{l1} = 6.06 m/ms, C_{t1} = 3.23 m/ms, ρ_1 = 4.46 g/cm^3 and for Alpha Case C_{l2} = 6.66 m/ms, C_{t2} = 3.553 m/ms, ρ_2 = 4.46 g/cm^3 .

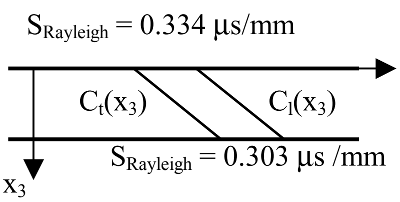

Figure 2 : Linear profile of the velocities.

A key feature of the calculated dispersion spectrum (Fig. 3) is that the fundamental branches have two different asymptotic limits [9] ($S_R = 0.334$ and $S_t(x_3 = 0) = 0.31$). The latter one, with the slowness only slightly exceeding $S_R = 0.334$, reveals itself at higher frequency than the range displayed in figure 3.

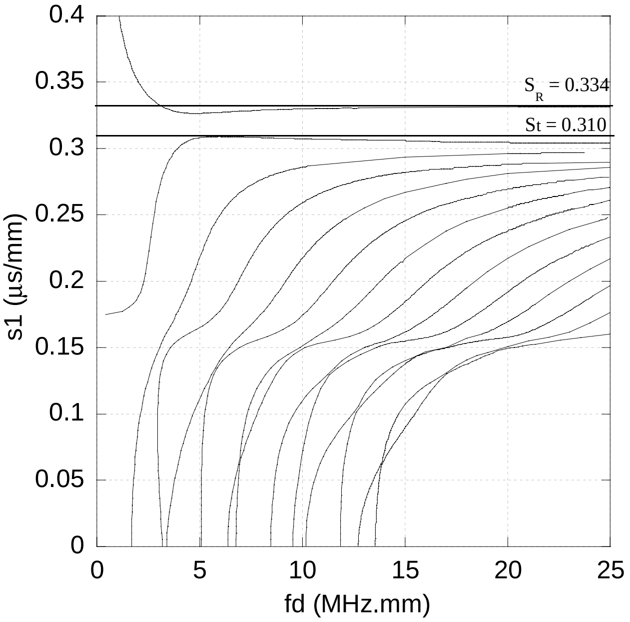

Figure 3 : P-SV dispersion curves for the linear profile shown in Fig. 2.

Computing the displacement field of the fundamental branches at intermediate frequency shows that the wave is localized closely to the plate faces, near the upper boundary for the flexural branch and near the lower face for the other fundamental branch before it breaks away towards $S_t = 0.31$.

Acoustic channels

For simulation of a more intricate profile, the pattern shown in Fig. 4 has been chosen. The density is kept at an arbitrary constant (ρ = 1g/cm^3). The profile is substantially varying and contains two minima, which are expected to entail plateaus and local asymptotic limits (acoustic channels) in the dispersion spectrum.

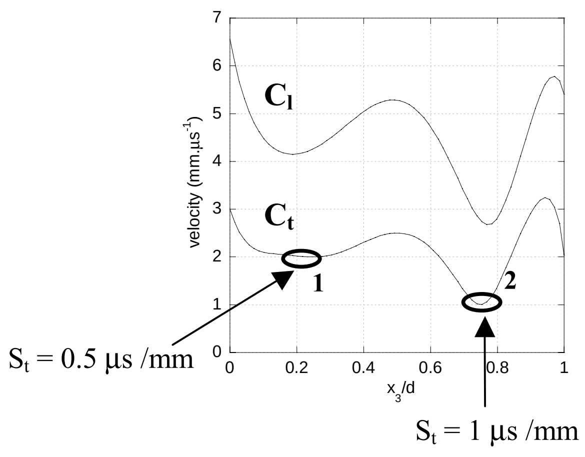

Figure 4 : Longitudinal and transverse velocity profiles.

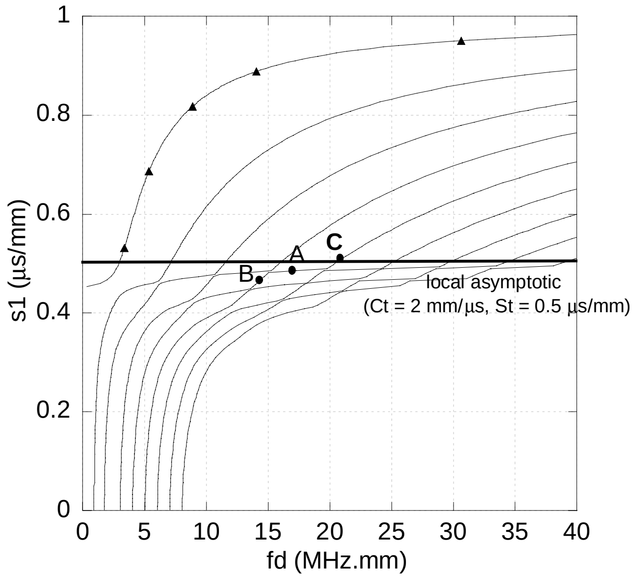

Figure 5 : SH dispersion curves for the profile shown in Fig. 4.

The dispersion curves for SH-waves present a specific network with a local asymptotic limit corresponding to the value of the local minimum for transversal velocity. The displacement at the point A on the plateau is confined to the range of the acoustic channel induced by the local minimum. Once there is a slight breakaway from the plateau (points B and C) the displacement field is almost zero in the range of the acoustic channel **1** (Fig. 4).

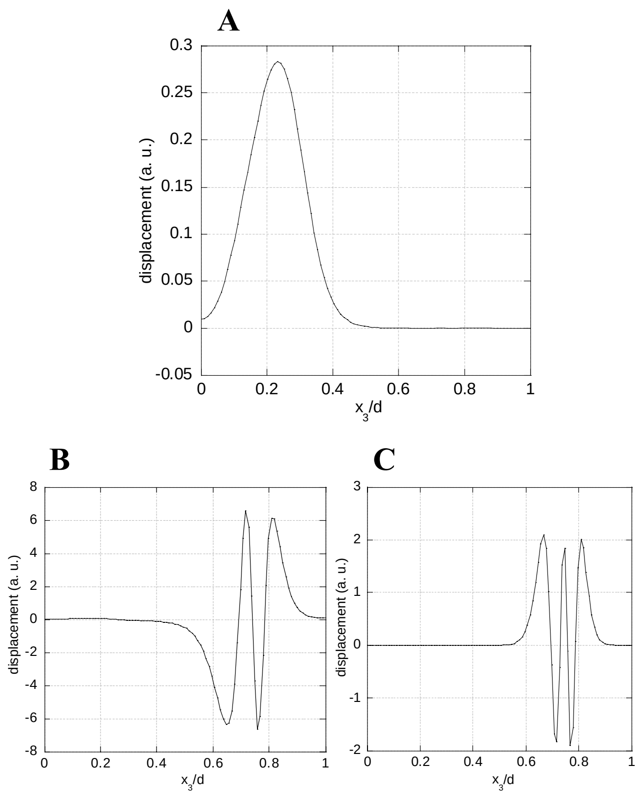

Figure 6 : SH-displacement fields.

These results show that the characteristic tendencies of the continuous profiles of material properties such as the minima of the transversal velocity are clearly detectable, by plotting the dispersion curves and the mode shapes of the ultrasonic waves propagating in the plate.

Conclusion

The Peano expansion of the matricant allows us to preserve the continuity of the material profiles, and to obtain a numerical approximation of an exact solution of an unaltered problem. This key feature of the method provides a plainly controllable degree of accuracy for the calculation. The results obtained show the potentiality of the method. The procedure of calculation in anisotropic cases remains exactly the same. Some further work has to be done on the determination of the degree of accuracy and on the improvement of the calculation at high frequency.

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