A REDUCED MODEL OF THE WAVE PROPAGATION IN SOLIDS WITH THIN COATINGS AND INNER LAYERS

D.D. Zakharov and L.J. Fradkin
FESBE, London South Bank University, 103 Borough Road, London SE1 0AA, UK
zakhard@lsbu.ac.uk

Abstract

A high order asymptotic model of the dynamic behaviour of solids with thin laminates is suggested. The main advantage of this approach consists in the essential reduction of the problem dimension when dealing with a reasonable frequency range. The respective algorithm is a shorter (and faster) alternative to the direct calculation of the wave spectra and diffraction phenomena for different applications in acoustics. For this purpose an approximate dynamic model of thin laminate under non-classical boundary conditions on its faces is deduced. The arbitrary number of plies and its layup, as well as the type of media and anisotropy, is considered. The internal relations between the values of stresses (or pressure) and of displacements on the laminate faces are derived. Three cases are investigated in details: solid substrate, covered by anisotropic elastic laminate; two solids with anisotropic elastic laminate in between; two solids with a thin intermediate fluid couplant.

Introduction

Despite the existence of direct methods to calculate the spectra of layered media [1-3] the simple and fast algorithm for complicated laminates is still an open question in view of problem dimension, stability, and qualitative analysis. There are a lot of pure numerical papers on different aspects of ultrasonic inspection (see, e.g. [4-6] with respective reviews) using direct methods. Those devoted to asymptotic approach are not so numerous [7-11] and their luck consists in the insufficient accuracy of model and cumbersome substitution of asymptotic series in global propagation matrix. In what follows the so-called asymptotic integration method [12-14] is used. It is well known as the powerful tool in the theory of thin plates and shells. In hydrodynamics some similar techniques was used in the theory of shallow water (e.g. [15,16]).

Foundations

To begin with obtain the relations between the values of displacements and stresses on the faces. Consider elastic laminate of arbitrary number of plies and their stacking sequences. Assume the perfect contact between plies. On the faces of laminate there are prescribed stresses or/and displacement, but at least one boundary condition must be formulated in terms of displacement. Each infinite ply should satisfy general 3D dynamic equations of elasticity with arbitrary material anisotropy and respective boundary conditions.

Suppose that the ratio $\varepsilon = h/l << 1$ where $H = 2h$ is a total thickness and l is a characteristic wavelength. The respective timescale of dynamic process has the order $O(\varepsilon^{\tau})$. Introduce Cartesian coordinates $(\mathbf{x}, x_3 \equiv z)$ and decompose the displacements $(\mathbf{U}, U_3 \equiv W)$ and stresses σ_{pq} into asymptotic ε-power series

$$\mathbf{d} = h\varepsilon^{\lambda}\left\{\mathbf{d}^0 + \varepsilon\mathbf{d}^1 + \ldots\right\},\ \mathbf{d} \equiv (W, U_2, U_1)^T.$$

It is easily to show that the only meaningful values are $\tau = 0$, $\lambda = -1$ and displacements and stresses in each layer satisfy the chain of recurrent relations

$$\partial_z^2 \mathbf{d}^s = \mathbf{G}_0^{-1}\left(\mathbf{A}\mathbf{d}^{s-2} - \mathbf{D}_1\mathbf{d}^{s-1}\right),$$

$$\mathbf{t}_z^s = \mathbf{G}_{\parallel}^T\mathbf{D}^T\mathbf{d}^{s-1} + \mathbf{G}_0\partial_z\mathbf{d}^s,$$

$$\mathbf{t}_x^s = \mathbf{G}_{\perp}^T\mathbf{D}^T\mathbf{d}^{s-1} + \mathbf{G}_*\partial_z\mathbf{d}^s,$$

where the dimensionless longitudinal coordinate $\mathbf{x}$ and transversal coordinate z are normalized over h and l, respectively. Here

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \partial_2 & \partial_1 \\ 0 & \partial_1 & \partial_2 & 0 & 0 \\ \partial_1 & \partial_2 & 0 & 0 & 0 \end{bmatrix},$$

and the stress vectors and operators

$$\mathbf{t}_z = \begin{bmatrix} \sigma_{zz} \\ \sigma_{2z} \\ \sigma_{1z} \end{bmatrix},\ \mathbf{t}_x = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix},$$

$$\mathbf{G}_0 = \mathbf{G}_{345}^{345},\ \mathbf{G}_* = \mathbf{G}_{345}^{162},\ \mathbf{G}_{\perp} = \mathbf{G}_{162}^{16245},$$

$$\mathbf{G}_{\parallel} = \mathbf{G}_{345}^{16245},\ \mathbf{D}_1 = \mathbf{D}\mathbf{G}_{\parallel} + \mathbf{G}_{\parallel}^T\mathbf{D}^T,$$

$$\mathbf{A} = \rho\partial_t^2 - \mathbf{D}\mathbf{G}_{16245}^{16245}\mathbf{D}^T$$

are dimensionless using reference values of Young's modulus, mass density and time. Matrix-minors are extracted from the stiffness matrix $\mathbf{G}_{123456}^{123456}$, which may have no zero blocks in each layer.

The careful analysis of leading terms using recurrent relations permits us to derive relations between the values of stresses $\mathbf{t}_z$ and displacements $\mathbf{d}$ on the upper and lower faces ($\mathbf{t}_z^{\pm}$ and $\mathbf{d}^{\pm}$, respectively) and their half sum and half difference

$$\mathbf{t}_\pm = \frac{1}{2}(\mathbf{t}_z^+ \pm \mathbf{t}_z^-),\ \mathbf{d}_\pm = \frac{1}{2}(\mathbf{d}^+ \pm \mathbf{d}^-).$$

When proceeding to the problem of solid with laminated coating or to two solids with laminate in between the scaling remains the same and timescale is $O(1)$. Thus, the obtained formulae give the necessary answer how to reduce dimension: to consider respective relations for $\mathbf{t}_z^\pm$ and $\mathbf{d}^\pm$ as so-called impedance boundary conditions (IBC) for two solids or for single solid.

Second order model for N-layered laminate

First, let us present the general dimensional results, which look as follows

$$\Delta\mathbf{t}_+ = 2\mathbf{d}_-,\ \mathbf{t}_- = 0 \text{ (the relative error is } O(\varepsilon));$$

$$\mathbf{\Delta t}_+ = \left\{2 + \sum h_j \mathbf{M}_j \left(\mathbf{\Delta}_j^+ - \mathbf{\Delta}_j^-\right)\mathbf{\Delta}^{-1}\right\}\mathbf{d}_- + \left(\sum h_j \mathbf{N}_j^T\right)\mathbf{d}_+,$$

$$\mathbf{t}_- = -\left(\sum h_j \mathbf{N}_j\right)\mathbf{\Delta}^{-1}\mathbf{d}_- \text{ (the error is } O(\varepsilon^2))$$

and coincide with the desired IBC for two solids with a laminate in between. Here h_j is a thickness of j th layer and other operators are

$$\mathbf{\Delta} = \sum h_j \mathbf{G}_{0j}^{-1},\ \mathbf{\Delta}_j^\mp \equiv \sum\nolimits_\mp h_k \mathbf{G}_{0k}^{-1},$$

$$\mathbf{N} = \mathbf{DG}_\parallel \mathbf{G}_0^{-1},\ \mathbf{M} = \mathbf{N} + \mathbf{N}^T$$

where summation $\Sigma_\mp$ assumes the sum over all indices smaller (larger) than j and the ply numbers begin from the lower face.

Second, transform these relations in the form convenient for single solid (half-space) with laminated coating: *under given stresses on its face* $z = z^+$

$$\mathbf{t}_z^- = \mathbf{t}_z^+ \text{ (the error is } O(\varepsilon)),$$

$$\mathbf{t}_z^- = \left\{1 + \left(\sum h_j \mathbf{N}_j\right)\right\}\mathbf{t}_z^+ \text{ (the error is } O(\varepsilon^2))$$

and *under given displacement on its face*

$$\mathbf{d}^- = \mathbf{d}^+ - \mathbf{\Delta t}_z^- \text{ (the error is } O(\varepsilon)),$$

$$\mathbf{d}^- = \left\{1 + \left(\sum h_j \mathbf{N}_j^T\right)\right\}\mathbf{d}^+$$

$$-\left\{\mathbf{\Delta} + \frac{1}{2}\sum h_j \left[\mathbf{M}_j\left(\mathbf{\Delta}_j^- - \mathbf{\Delta}_j^+\right) + \mathbf{N}_j^T\mathbf{\Delta} - \mathbf{\Delta N}_j\right]\right\}\mathbf{t}_z^-$$

(the error is $O(\varepsilon^2)$),

respectively. In a similar way the relations of high order can be derived. For the sake of shortness let us represent principal results for a single layer.

High order model for a single layer

General relations of the order $O(\varepsilon^6)$ acquire the form

$$\mathbf{d}_- = h\mathbf{G}_0^{-1}\mathbf{t}_+ + \mathbf{G}_0^{-1}\left\{-h\mathbf{G}_0\mathbf{N}^T + \frac{h^3}{3}\mathbf{D}_1\mathbf{G}_0^{-1}\mathbf{A} - \right.$$

$$-\frac{h^5}{6}\left[\mathbf{D}_1\mathbf{G}_0^{-1}\left(\mathbf{A} - \frac{1}{5}\mathbf{B}\right) + \left(\frac{1}{3}\mathbf{B} - \frac{1}{5}\mathbf{A}\right)\mathbf{G}_0^{-1}\mathbf{D}_1\right]\left.\mathbf{G}_0^{-1}\mathbf{A}\right\}\mathbf{d}_+ +$$

$$+\mathbf{G}_0^{-1}\left\{-\frac{h^2}{3}\mathbf{B} + \frac{h^4}{5}\left[\frac{2}{3}\mathbf{D}_1\mathbf{G}_0^{-1}\mathbf{AG}_0^{-1}\mathbf{D}_1 + \frac{1}{4}\mathbf{BG}_0^{-1}\mathbf{B}\right]\right\}\mathbf{d}_-,$$

$$(\mathbf{B} \equiv \mathbf{A} + \mathbf{D}_1\mathbf{G}_0^{-1}\mathbf{D}_1) \qquad (1)$$

$$\mathbf{t}_- = \left\{-\mathbf{NG}_0 + \frac{h^2}{3}\mathbf{AG}_0^{-1}\mathbf{D}_1 - \frac{h^4}{5}\mathbf{AG}_0^{-1}\left[\frac{2}{3}\mathbf{AG}_0^{-1}\mathbf{D}_1 + \right.\right.$$

$$\left.\left.+\frac{1}{4}\mathbf{D}_1\mathbf{G}_0^{-1}\mathbf{B}\right]\right\}\mathbf{d}_- + \mathbf{A}\left[h - \frac{h^3}{3}\mathbf{G}_0^{-1}\mathbf{A}\right]\mathbf{d}_+,$$

and may be rewritten again for a coated solid as

$$\mathbf{t}_- + h\left\{\mathbf{N} \pm h\mathbf{L}_2 + h^2\mathbf{S}_3 \pm h^3\mathbf{R}_4 + h^4\mathbf{P}_5\right\}\mathbf{t}_+ -$$

$$-2h\left\{\mathbf{L} \pm h\mathbf{LN}^T + h^2\mathbf{M}_4 \pm h^3\mathbf{Q}_5\right\}\mathbf{d}^\pm = 0, \quad (2)$$

or

$$\mathbf{d}_- + h\left\{\mathbf{N}^T \pm h\mathbf{L}_2^T + h^2\mathbf{F}_3^T \pm h^3\mathbf{H}_4^T + h^4\mathbf{J}_5^T\right\}\mathbf{d}_+ -$$

$$-h\mathbf{G}_0^{-1}\left\{1 \pm h\mathbf{N} + h^2\mathbf{N}_2 \pm h^3\mathbf{R}_3 + \right.$$

$$\left.+h^4\mathbf{T}_4 \pm h^5\mathbf{Y}_5\right\}\mathbf{t}_z^\pm = 0 \qquad (3)$$

where operators are expressed via those from general formulae. Formulae (2) and (3) correspond to the problems of *given stresses* and *given displacements* on the coating surface.

Fluid coupled solids

The main difference consists in the type of the layer medium. Let us investigate two cases: compressible and incompressible fluid. Inside the layer 3D equations of fluid dynamics hold, on the interface the transversal displacement must be continuous, the normal stress should satisfy the equation of pressure balance and the tangent stresses are absent (when dealing with inviscid fluid). All this leads to the slightly changed asymptotic series for the displacement field in fluid

$$W = h\varepsilon^\lambda\left(w^0 + \varepsilon w^1 + \ldots\right),$$

$$U_\alpha = h\varepsilon^{\lambda+1}\left(u_\alpha^0 + \varepsilon u_\alpha^1 + \ldots\right),$$

and for the naturally introduced displacement potential in the Newtonian *compressible* fluid

$$\Psi = h^2\varepsilon^\lambda\left(\psi^0 + \varepsilon\psi^1 + \ldots\right).$$

The corresponding recurrent formulae are obtained from the equation of motion

$$\partial_z^2\psi^l = A\psi^{l-2},\ A \equiv \partial_t^2 - \nabla^2$$

the expressions of displacements and pressure p

$$w = \partial_z\psi^l,\ u_\alpha^l = \partial_\alpha\psi^l,\ p^l = -\partial_t^2\psi^l$$

and the pressure balance on the interfaces

$$p^{\pm,l-2-\lambda} = -\sigma_z^{\pm,l} + (\rho + \rho_\pm)g'w^{\pm,l-1-\lambda}. \qquad (4)$$

Here stresses and pressure are normalized over ρc^2 (ρ is a fluid mass density, c is its sound speed) and last term in (4) is responsible for the contribution of gravitation

$$g' = \frac{gL}{\rho c^2}\ , \qquad \mathrm{g} \approx 9.8\mathrm{m/sec}^2 .$$

The procedure of asymptotic integration is much simpler than for solid layers and the following 2D dimensional IBC of the order $O\left(\varepsilon^{10}\right)$ are derived finally

$$-\sigma_{zz}^{+} + (\rho_{+} + \rho) g W^{+} + \sigma_{zz}^{-} - (\rho_{-} + \rho) g W^{-} =$$
$$= -h\rho\partial_t^2 \left\{ 1 - \frac{h^2}{3} B + \frac{2h^4}{15} B^2 - \frac{17h^6}{315} B^3 \right\} \left(W^{+} + W^{-} \right),$$
$$(B \equiv c^{-2}\partial_t^2 - \nabla^2) \qquad (5)$$
$$hB\left\{ -\sigma_{zz}^{+} + (\rho_{+} + \rho) g W^{+} - \sigma_{zz}^{-} + (\rho_{-} + \rho) g W^{-} \right\} =$$
$$= -\partial_t^2 \left\{ 1 + \frac{h^2}{3} B - \frac{h^4}{45} B^2 + \frac{2h^6}{945} B^3 - \frac{1343h^8}{38102400} B^4 \right\} \left(W^{+} - W^{-} \right).$$

They are completed by the condition of zero tangent stresses.

In the case of *incompressible* fluid there are a few technical changes but the final dimensional IBC look quite similar with the only replacement $B \equiv -\nabla^2$.

Numerical tests

Since the partial waves are basic elements for any spectral problem the simplest calculations of the response to incident P- or S-wave in elastic half-space are reasonable to characterize the numerical accuracy of model. To be brief let us represent just a few media.

Table 1 : Parameters of the media.

Types of media	c_P, c_S or c [m/sec]		ρ [kg/m^3]
Polystyrene(Po)	2350	1150	1060
Aluminum (Al)	6320	3080	2700
Epoxy (Ep)	2800	1100	1300
Water (Wa)	1400		1000

The response to incident wave from the first half-space gives rise two reflected waves and two transmitted waves (when dealing with two solids). Their complex magnitudes are calculated by the exact method using propagation matrix and by approximate asymptotics using relations (1)-(3) and (5). In Figures 1-4 the typical relative mean error is shown for two solids (Al-Ep-Po), coated half-space Al-Po (substrate) with stress free face and Al-Ep (substrate) with clamped face, and for the fluid coupled solids (Po-Wa-Al, water presumed to be compressible). The angle of incidence is measured from the normal to the interface.

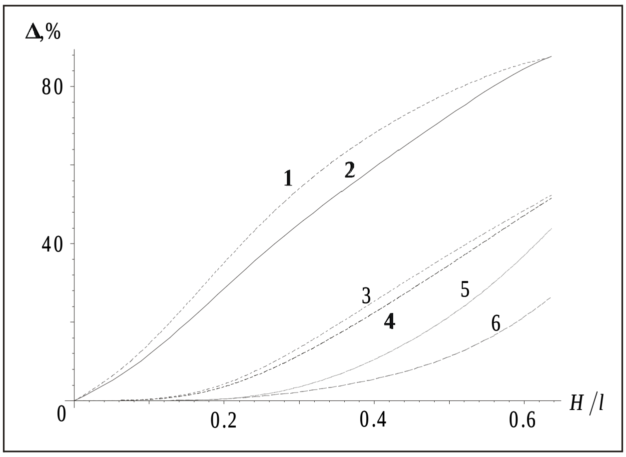

Figure 1 : Al-Ep-Po; angle of incident P-wave is 50^o

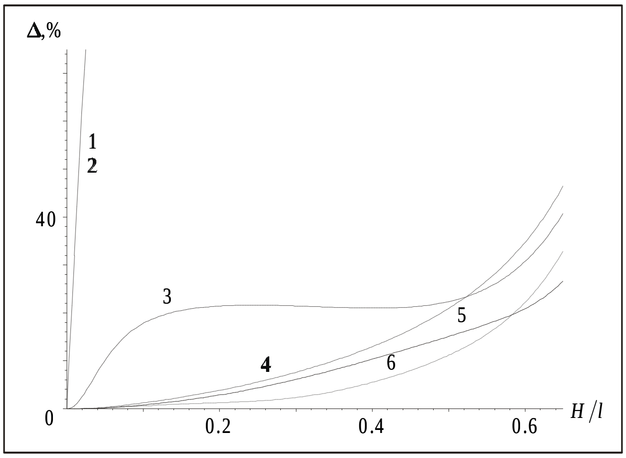

Figure 2 : Al-Po; angle of incident S-wave is 10^o

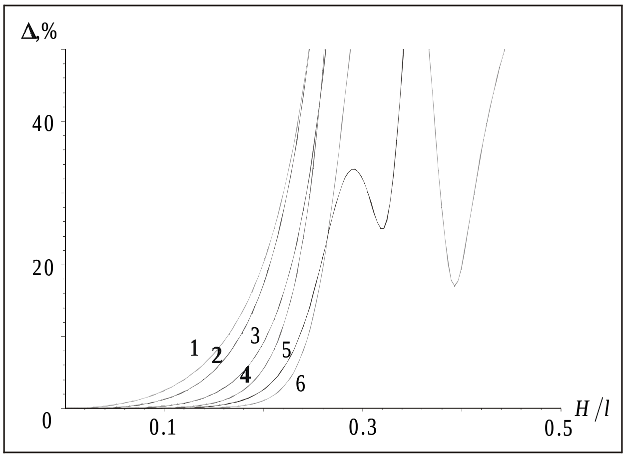

Figure 3 : Al-Ep; angle of incident P-wave is 10^o

As one can see for solid coupled solids our results agree well with exact ones when

$$H/l < 0.2 \div 0.25 .$$

The natural restriction for us is the first quasi resonance frequency of thin layer.

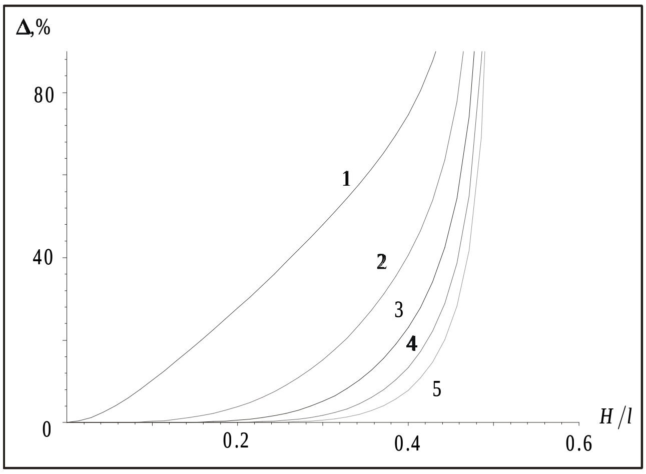

Figure 4 : Po-Wa-Al; angle of incident S-wave is 10^o

As far as fluid is concerned the accuracy of model is higher and presented approximation is valid when $H/l < 0.3 \div 0.35$. Since the media are very contrast the first quasi resonance frequency of fluid layer is $H/l \approx 0.5$. The case of incompressible fluid is also calculated and the interval of model applicability there is about two times larger then for the compressible fluid.

Conclusion

The general asymptotically justified 2D high order approximate model of solids with thin coating or inner laminate is derived. It is applicable for stratified structural members and fluid coupled solids. As shown the model is low frequency with respect to the thin laminates but may be high frequency with respect to the main thick solids.

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