MODELLING OF THE PLATE FLEXURAL EDGE MODE USING REFINED ASYMPTOTICS

D.D. Zakharov
Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1 Vernadsky Avenue, Moscow 113526, Russia
dd_zakh@mail.ru

Abstract

In sixtieth the Rayleigh-type waves of bending nature (RTWBN) have been considered in isotropic plates using 2D Kirchhoff's plate theory. Since then they remain attractive for ultrasonic applications but are still poorly investigated. During last years RTWBN are revealed in anisotropic media as well. The contribution presented is an attempt to clarify the RTWBN nature in isotropic plate and its sensibility to the mathematical tool required what is the main topic of this contribution. As shown, the description of leading part of RTWBN can be essentially simplified and its wave speed is calculated analytically. The results agree with those of experiments and FEM calculations.

Introduction

There are a lot of papers and at least two monographs whose authors considered RTWBN. Many of them investigated these waves on the basis of Kirchhoff's theory [1-3] using classical boundary conditions, despite the fact that equations and boundary conditions are of different accuracy there. Thus, the more accurate consideration involves both high order equations and modified boundary conditions. The attempt to use Timoshenko-Reissner-Mindlin theory (which is asymptotically not justified) has been performed in [4] with comparison to the results of experimental measurement and FEM calculations [5,6]. In this paper the correct RTWBN model using refined Kirchhoff's plate theory is used and the influence of correct boundary conditions is studied. Upon this estimate the high order equations are used to describe the leading part of desired RTWBN with respective explanation why it works and what is essential in the nature of these waves. Finally, the wave speed can be calculated using two simple analytical expression and the results are in perfect agreement with experiment.

Influence of boundary conditions: Kirchhoff's theory

Consider a semi-infinite elastic plate with a total thickness $H = 2h$, made of isotropic material with Young modulus E, Poisson's ratio ν and mass density ρ, and occupying a region $x_2 \geq 0$, in its middle plane (see Figure 1). From the asymptotical viewpoint when the plate is thin, i.e. $\varepsilon = h/L << 1$ where L is a longitudinal scale (wavelength), and when timescale $T = O(\varepsilon^{-1})$, the normal deflection w is sought in the form of asymptotical power series

$$w = L\varepsilon^{-3}\left(w^0 + \varepsilon w^1 + \ldots\right). \qquad (1)$$

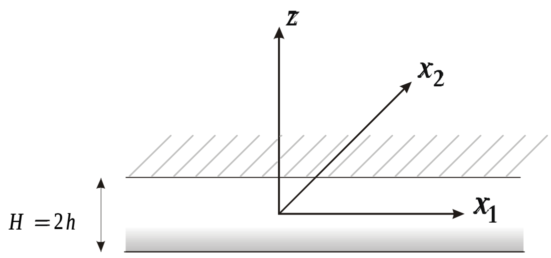

Figure 1 : Plate geometry

According to the usual Kirchhoff's theory the normal deflection $w = w(\mathbf{x}, t)$ and satisfies the equation

$$\left\{2\rho h\partial_t^2 + d\Delta^2\right\}w = 0, \quad \left(\Delta \equiv \partial_1^2 + \partial_2^2\right) \qquad (2)$$

$$d = \frac{2Eh^3}{3\left(1-\nu^2\right)},$$

and boundary conditions on the stress free edge in the form of normal stress couple and Kirchhoff's shear force absence

$$M_{22} = 0, \; P_{2z} = 0. \qquad (3)$$

When seeking a free time-harmonic RTWBN

$$w = Ae^{i(\omega t - k_1 x_1) - k_2 x_2}, \; k_1 > 0$$

the respective value of dimensionless wave speed

$$s \equiv \sqrt[4]{\frac{2\rho h\omega^2}{dk_1^4}} = \frac{V_K}{V_B}, \; V_K \equiv \frac{\omega}{k_1}, \; V_B = \sqrt[4]{\frac{d\omega^2}{2\rho h}}$$

has been obtained first in [1] and equals to

$$s_* = \sqrt[4]{(1-\nu)\left(-1 + 3\nu + 2\sqrt{1 - 2\nu + 2\nu^2}\right)}, \qquad (4)$$

$$s_* = 1 - \frac{\nu^4}{16} - \frac{\nu^5}{8} - \frac{5\nu^6}{32} + O(\nu^7).$$

But due to (1) the error of equation (2) is $O(\varepsilon^2)$ and the error of (3) is $O(\varepsilon)$. The question of interest is the influence of the modifications of boundary conditions. As follows from the boundary layer analysis [7] the second order modification of boundary conditions (1) acquires the form

$$M_{22} + \chi h \partial_1 M_{12} = 0,\ \chi = \sum_{n=1}^{\infty} (2n-1)^{-5} \approx 1.260497.$$

The dispersion equation $F(s^2, \sigma, \nu) = 0$, where $\sigma = \chi k_1 h \geq 0$, always has a unique root $s(\sigma, \nu) \in (0,1)$. The plot of this function is shown in Figure 2. As one can see in the origin $s = s_*$ and $s \to 1$ for large σ, so the difference $s - s_*$ is less than 1%.

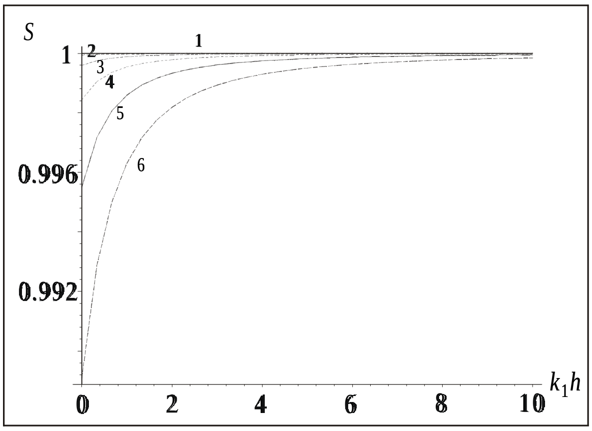

Figure 2 : Roots s at different Poisson's ratio $\nu = 0, 0.8(3), 0.1(6), 0.25, 0.3(3), 0.41(6), 0.5$ (curves 1,2,…6, respectively).

Since ν^4 is small quantity the expansion of s into power series with respect to σ

$$s = s_* + s_1\sigma + s_2\sigma^2 + O(\sigma^3),$$

$$s_1 = \frac{\nu^4}{8\sqrt{2}}\left(1 + \frac{\nu}{2} + \frac{9\nu^2}{4} + \frac{3\nu^3}{4} - \frac{87\nu^4}{32} + O(\nu^5)\right),$$

$$s_2 = -\frac{3\nu^4}{32}\left(1 + 2\nu + 2\nu^2 - \frac{2\nu^3}{3} - \frac{77\nu^4}{12} + O(\nu^5)\right),$$

explains the low sensibility of root to the perturbation of boundary conditions. But the influence of the main differential operator is essential.

High order equations of plate bending

Further improvement in the plate modelling meet considerable technical obstacles caused by taking into account both the anti-plane boundary layers and the plane one for the boundary conditions of high-order iterations. At the meantime the refinement of the homogeneous Kichhoff equation may be easily done. The equation of plate bending with the error $O(\varepsilon^8)$ is deduced using 3D elasticity [8] and recurrent formulas for next terms of asymptotic expansions (1), and is reduced to a final form in terms of normal deflection w

$$2\rho h\left\{1 + a_0(\nu)h^2\Delta + a_1(\nu)\frac{h^2}{c_2^2}\partial_t^2 + a_2(\nu)\frac{h^4}{c_2^2}\partial_t^2\Delta\right\}\partial_t^2 w + \\ + d\Delta^2 w = 0, \qquad (5)$$

where c_2 is shear wave speed and $a_k(\nu)$ are coefficients. The evident analogue with equation (2) is just in the correction of term of inertia. For a free plate vibration mode, which satisfies the condition $\Delta w < 0$, on expressing Δw upon this analogue from (5) and on substituting again into (5) the high order equation of a plate in terms of *effective inertia* [8] yields

$$\left\{d\Delta^2 - 2\rho h\omega_n^2\right\}w = 0\,,\ \omega_n^2 = \omega^2 \sum_{k=0}^{n} B_k(\nu)\left(\frac{\omega h}{c_2}\right)^k,$$

where $B_k(\nu)$ are coefficients. For indices $n = 0,1,2,3$ the relative truncation error of the respective high-order operator is $O(\varepsilon^{2n+2})$. The *assumption* $\Delta w < 0$ corresponds to many practically important cases. In particular, it holds for the propagation vibration modes. For the latter, correction of the equation of motion seems to be more important then that of boundary conditions. As shown below the last feature may also characterise the most important components of RTWBN because they obey the same assumption.

Wave speed description

From the above follows the efficient way to calculate the speed of RTWBN. The Kirchoff's theory gives us two partial waves and one of them with smaller attenuation is responsible for the localisation. It also satisfies the inequality $\Delta w < 0$. In what follows we deal with this partial wave and with the respective root only.

In accordance with (2) and (5) one can see that the characteristic root of equations is the same but the meaning of root for equation (5) differs

$$s^4 \equiv \frac{2\rho h\omega_n^2}{dk_1^4} = \left(\frac{V_K}{V_B}\right)^4 \sum_{k=0}^{n} B_k(\nu)\left(\frac{\omega h}{c_2}\right)^k \qquad (6)$$

at different indices n. In addition one may set $s \approx s_*$ given by (4) in the right hand side. Then, the wave speed ratio is expressed from (6) and improved at $n = 0,1,2,3$. Finally we arrive at the complete description of the wave speed using two analytical formulae (4) and (6).

Comparison with experiment

Compare our results with those obtained by P. Lagasse and F. Oliner [5,6]. Their results of the wave speed measurement are in a good agreement with FEM calculation. To use the same scale let us introduce the dimensionless frequency $\Omega = \omega H / c_R$, where c_R is the Rayleigh wave speed, and transform the equation (6) as follows

$$\Omega\sqrt{\sum_{k=0}^{n} B_k\left(\frac{\Omega\psi}{2}\right)^k} = \sqrt{\frac{8}{3(1-\nu)}}\frac{(sk_1h)^2}{\psi}, \ \psi = \frac{c_R}{c_2}.$$

The results of calculation for Poisson's ratio $\nu = 0.39$ are shown in Figure 3. The letter **K** marks the curve for classical Kirchhoff's theory, $\mathrm{K_R}$ denotes the same with refined boundary conditions ($n = 0$), and numbers 1,2,3 correspond to the iteration indices in formula (6). Balls represent experimental measurement of the wave speed [6]. As seen, there is only a slight difference between the curves **K** and $\mathrm{K_R}$ (practical coincidence), i.e. these results are now asymptotically justified within the second order asymptotic error. Iteration $n = 1$ gives the intermediate results but when using two iterations ($n = 2$) the wave speed matches well to the experimental data. The result of third order iteration ($n = 3$) agrees with experiment quite good. The additional curve, marked by **T**, is obtained in [4] using Timoshenko-Reissner-Mindlin theory of plate. It slightly deviates from the curve 3, especially at large frequencies where we hope to deal with a better approximation. Let us discuss this fact now.

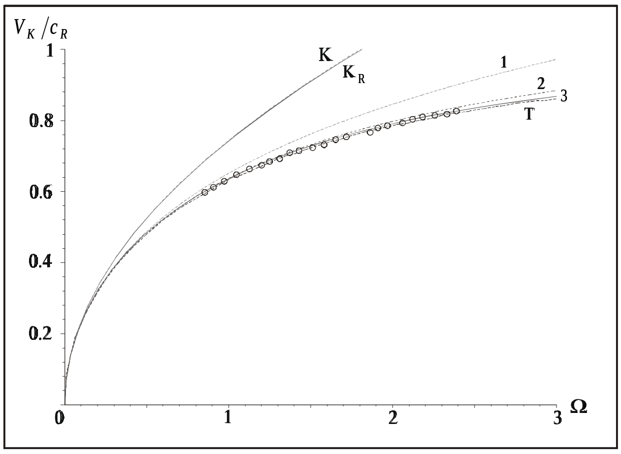

Figure 3 : Comparison of numerical results

What and why?

As shown the description obtained gives result of good quality despite the fact, that a very restricted tool is used. Namely, just the iteration of the main operator without changing boundary conditions and substitution of the root for one partial wave leads to this. To understand the matter let us imagine the correct high order model – both operator and boundary conditions. A set of characteristic roots contain two, which are modified roots of Kirchhoff's model, and other "parasitic" ones, which describe solution of *high variation* near the edge and *high attenuation* when far from the edge. By the way they are a bit contradictive, since the initial long wave asymptotic assumption demands a *slow* variation and a large longitudinal scale L. They must be very sensible to the modification of boundary conditions, but the less attenuated component is low sensible to it. And above we improve exactly this partial wave.

As far as Timoshenko-Reissner-Mindlin (TRM) theory is concerned let us have a look at the common and at the difference with the above. Assumptions of TRM differ from Kirchhoff's theory and the additional rotation inertia is involved. After simplifications the main operator can be rewritten in the form

$$2\rho h\left\{1 + m_0(\nu)h^2\Delta + m_1(\nu)\frac{h^2}{c_2^2}\partial_t^2\right\}\partial_t^2 w + d\Delta^2 w = 0,$$

similar to (5) but of less order and with different coefficients. The respective boundary conditions acquire the form

$$M_{12} = 0, \ M_{22} + \chi h\partial_1 M_{12} = 0, \ P_{2z} = 0,$$

which also contains one term, different from Kirchhoff's theory. So, it is not asymptotically justified. By the way, Reissner has shown [9] that in the long-wave (and low frequency) limit of his theory these three boundary conditions are reduced to two

$$M_{22} + \chi_* h\partial_1 M_{12} = 0, \ P_{2z} = 0$$

and $\chi_* \approx 1.264$ is close to χ. The happiness of TRM theory is in its good description of low frequency phenomena as well as in high frequency behaviour of Rayleigh asymptotics of plate modes. Its "parasitic" roots interacts happily with physically meaningful roots arrived from classical Kirchhoff's theory.

Conclusion

The performed analysis exhibits some essential properties of the edge bending waves in a thin isotropic plate. *First*, it concerns the possibility to describe this wave at relatively high frequencies where the average 2D plate theory is used very seldom. *Second*, it clarifies the nature of this wave. The leading part mostly responsible for the wave localisation can be singled out. Its behaviour is determined by the leading part of the differential equation of plate bending and then it is improved when taking account of the highest order operators. Its sensibility to the improvement of the boundary condition can be neglected. Other partial waves arisen in this representation must be very sensible to the perturbation of boundary conditions. *Third*, such an important acoustic characteristic as wave speed may be predicted very easy with high accuracy by formulae (4) and (6).

Acknowledgements

This work has been carried out under partial support from the Russian Foundation for Basic Research (grant 02-01-00843), which is gratefully acknowledged. The author is also very thankful to the organizers of WCU in Paris.

References

[1] Yu. K. Konenkov, "A Rayleigh-type flexural wave," Sov. Phys. Acoust., vol. 6, pp. 122-123, 1960.
[2] R. N. Thurston, J. McKenna, "Flexural acoustic waves along the edge of a plate," IEEE Trans. Sonics Ultrasonics, vol. 21, pp. 296-297, 1974.
[3] B. K. Sinha, "Some remarks on propagation characteristics of ridge guide for acoustic waves at low frequencies," J. Acoust. Soc. America, vol. 56, pp. 16-18, 1974.
[4] A. N. Norris, V. V. Krylov, I. D. Abrahams, "Flexural edge waves and Comments on 'A new bending wave solution for the classical plate equation," J. Acoust. Soc. America, vol. 107, pp. 1781-1784, 2000.
[5] P. E. Lagasse, "High-order finite-element analysis of topographic guides supporting elastic surface waves," J. Acoust. Soc. America, vol. 53, pp. 1116-1122, 1973.
[6] P. E. Lagasse, A. A. Oliner, "Acoustic flexural mode on a ridge of semi-infinite height," Electron Lett., vol. 12, pp. 11-13, 1976.
[7] A. V. Kolos, "Methods of refining the classical theory of bending and extension of plates," J. Appl. Math. Mech. (PMM), vol. 29, pp. 771-781, 1965.
[8] A. L. Goldenveizer, J. D. Kaplunov, E. V. Nolde, "On Timoshenko-Reissner type theories of plates and shells," Int. J. Solids Structures, vol. 30, pp. 675-694, 1993.
[9] E. Reissner, "On the analysis of first and second order shear deformation effects for isotropic elastic plates", J. Acoust. Soc. America, vol. 47, pp. 959-961, 1980.