THE ROLE OF ANISOTROPY IN ACOUSTICS OF CRYSTALS

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Abstract
The paper presents a short review of some basic theoretical results in the acoustics of anisotropic media. It includes: the general theorems related to phase speeds and polarization vectors of three bulk-wave eigen-modes in arbitrary elastic media; the topological classification of polarization singularities for plane waves; the conditions for the occurrence of the phenomenon of energy concentration; the general criteria for the existence of different classes of surface acoustic waves in media with different couplings; and some other topics.
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Introduction

The role of anisotropy in crystalloacoustics is not at all reduced just to variations of wave characteristics for different directions of propagation. Anisotropy creates also qualitatively new properties of elastic waves and acoustic phenomena that have not got close analogues in isotropic media. Some of them have already found their practical applications in real devices.

A theoretical description of elastic waves in anisotropic materials is a very non-trivial problem. Impermeability of the wave equations for media of unrestricted anisotropy to explicit analysis has required development of new theoretical methods that allowed obtaining final conclusions without hopeless direct calculations. As a result, during the last few decades, due to contributions of many researchers from different countries, the theory of elastic waves in anisotropic media gradually became an independent branch of modern crystalloacoustics.

Elastic bulk waves

Some basic concepts

Consider an infinite elastic medium of unrestricted anisotropy in which a plane bulk wave of displacements $\mathbf{u}(\mathbf{x},t)$ is propagating along the direction specified by a unit vector $\mathbf{m}$,

$$\mathbf{u}(\mathbf{x},t) = u_0\mathbf{A}\exp[ik(\mathbf{m}\cdot\mathbf{x}-ct)]. \qquad (1)$$

Here u_0 is the scalar amplitude, $\mathbf{A}$ is the unit polarization vector, k is the length of the wave vector, $\mathbf{k}=k\mathbf{m}$, c is the phase speed, $c=\omega/k$. For each direction $\mathbf{m}$ it is generally possible a propagation of three independent *isonormal* waves with mutually orthogonal polarizations $\mathbf{A}_\alpha$ (α = 1, 2, 3) and generally different phase speeds c_α, which are determined from the eigenvalue problem (the *Christoffel equation*)

$$(mm)\mathbf{A}_\alpha = \rho c_\alpha^2\mathbf{A}_\alpha, \qquad (2)$$

where $(mm) \equiv \mathbf{Q}$ is the so-called *acoustical tensor* with components $(mm)_{jk} = m_i c_{ijkl} m_l = Q_{jk}$, ρ is the density of the medium and c_{ijkl} is its elastic moduli tensor.

In isotropic bodies there are only two independent (and certainly spherical) phase speed sheets $c_\alpha(\mathbf{m})$: the degenerate sheet $c_1 = c_2 = c_t = (\mu/\rho)^{1/2}$ and the separate sheet $c_3 = c_l = (\lambda/\rho)^{1/2}$, where μ and λ are the Lame coefficients. The latter relates to longitudinal waves polarized along the wave normal: $\mathbf{A}_l \parallel \mathbf{m}$, and the first one – to transverse waves arbitrarily polarized in the plane orthogonal to $\mathbf{m}$: $\mathbf{A}_t \perp \mathbf{m}$.

In crystals, normally none of the three isonormal waves is purely transverse or longitudinal. And their speeds c_α are generally non-degenerate. However along some specific directions $\mathbf{m}$ even in triclinic crystals it is possible propagation of transverse or longitudinal or degenerate elastic waves. Such special directions are called respectively *transverse* and *longitudinal normals* ($\mathbf{m}_t$ and $\mathbf{m}_l$) and *acoustic axes* ($\mathbf{m}_d$). As was proved by Fedorov [1], the equation

$$\mathbf{m}_t\mathbf{Q}^2\cdot[\mathbf{m}_t\mathbf{Q}\times\mathbf{m}_t] = 0 \qquad (3)$$

determines lines of solutions for transverse normals $\mathbf{m}_t$ on the unit sphere $\mathbf{m}\cdot\mathbf{m}$ = 1. Intersections of such lines must occur along longitudinal normals $\mathbf{m}_l$, which satisfy the equation [1]

$$\mathbf{m}_l \times \mathbf{Q}\mathbf{m}_l = \mathbf{0}. \qquad (4)$$

Kolodner [2] has shown that for any crystal this equation always has at least three solutions. On the other hand, in any crystal, apart from a transverse isotropic one, the number of solutions of Eqn.(4) can not exceed 13. In particular, exactly 13 longitudinal normals always exist in cubic crystals. For a transverse isotropic media Eqn. (4) is satisfied by any direction in the basal plane and for the wave normals belonging to a cone and making an angle

$$\theta_l = \tan^{-1}\left(\frac{c_{33}-2c_{44}-c_{13}}{c_{11}-2c_{44}-c_{13}}\right)^{\frac{1}{2}} \qquad (5)$$

with the principal axis, which is also a longitudinal normal. The detailed analysis of numbers of longitudinal normals admitted by various symmetry systems has been accomplished by Khatkevich [3, 4], Brugger [5] and Bestuzheva & Darinskii [6].

The third type of special directions in crystals – acoustic axes $\mathbf{m}_d$ – related to degeneracy of the phase speeds of a pair of isonormal waves will be considered separately. Here we shall just notice that along $\mathbf{m}_d$ any vector $\mathbf{A}$ orthogonal to the polarization $\mathbf{A}_3$ of the non-degenerate wave can be a polarization for the degenerate wave propagating with the speed $c_1 = c_2$.

Some general theorems on phase-speed branches

The determination of the basic wave parameters, the polarization vectors $\mathbf{A}_\alpha(\mathbf{m})$ and the phase speeds $c_\alpha(\mathbf{m})$, requires a solution of the eigenvalue problem (2), which reduces to a cubic secular equation

$$\det[\mathbf{Q} - \rho c_\alpha^2 \mathbf{I}] = 0. \tag{6}$$

Of course, it would be practically impossible to analyze this equation explicitly for arbitrary anisotropy. Nevertheless, as we shall see, one can extract a series of fundamental general properties of the phase speed branches $c_\alpha(\mathbf{m})$ without explicit solving Eqn. (6). Below we shall number the solutions of Eqn. (2) so that $c_1(\mathbf{m}) \le c_2(\mathbf{m}) \le c_3(\mathbf{m})$, calling the $c_1(\mathbf{m})$, $c_2(\mathbf{m})$ and $c_3(\mathbf{m})$ functions the *bottom*, *middle* and *top* branches of the phase speeds, respectively.

We start from the invariance properties of the combination $c_1^2 + c_2^2 + c_3^2$ on the sphere $\mathbf{m}^2 = 1$. Let $\mathbf{n}$, $\mathbf{s}$ and $\mathbf{t}$, be three mutually orthogonal unit vectors. As was proved in [7-9], the identity is valid

$$\sum_{\alpha=1}^{3} c_\alpha^2(\mathbf{n}) + \sum_{\alpha=1}^{3} c_\alpha^2(\mathbf{s}) + \sum_{\alpha=1}^{3} c_\alpha^2(\mathbf{t}) = (c_{11} + c_{22} + c_{33} + 2c_{44} + 2c_{55} + 2c_{66})/\rho = \text{const} \tag{7}$$

irrespective of the orientation of the system $\{\mathbf{n},\mathbf{s},\mathbf{t}\}$.

In the same paper [9] Alshits & Lothe have established a series of other useful properties of the branches $c_\alpha(\mathbf{m})$. For instance, for any propagation directions $\mathbf{m}$ and $\mathbf{n}$,

$$c_1(\mathbf{n}) \le c_3(\mathbf{m}). \tag{8}$$

This means that

The largest value of the phase speed in the bottom branch is not greater than the smallest value of the phase speed in the top branch. (9)

Theorem (9) leads to strict limitations on the possible configurations of sheets of the phase velocity surface. In particular, according to this theorem the whole inner sheet must be inside of the sphere of the radius $c_3^{\min} = \min\{c_3(\mathbf{m})\}$.

In [9] there were also established the following general relations between the phase speeds of waves traveling along the directions $\mathbf{m}$ and $\mathbf{A}_{1,3}(\mathbf{m})$:

$$c_1(\mathbf{m}) \ge c_1(\mathbf{A}_1), \qquad c_3(\mathbf{m}) \le c_3(\mathbf{A}_3). \tag{10}$$

In other words,

The polarization vector $\mathbf{A}_{1,3}$ of a wave moving along $\mathbf{m}$ at speed $c_{1,3}(\mathbf{m})$ and belonging to the bottom / top branch points out another propagation direction where the wave speed $c_{1,3}(\mathbf{A}_{1,3})$ does not exceed / is not less $c_{1,3}(\mathbf{m})$. (11)

Pursuing along this line we obtain an algorithm for the search for a minimum, $c_1^{\min} = \min\{c_1(\mathbf{m})\}$, and maximum, $c_3^{\max} = \max\{c_3(\mathbf{m})\}$.

Now let us consider the degeneracy between the bottom and middle branches in the direction $\mathbf{m}_d$, $c_1(\mathbf{m}_d) = c_2(\mathbf{m}_d)$. At the degeneracy point the polarization vector $\mathbf{A}$ can be expressed in the form

$$\mathbf{A} = \alpha \mathbf{A}_1 + \beta \mathbf{A}_2, \qquad \alpha^2 + \beta^2 = 1, \tag{12}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are arbitrary orthogonal unit vectors in the plane perpendicular to the polarization vector $\mathbf{A}_3(\mathbf{m}_d)$. One can prove that

$$c_1(\mathbf{m}_d) \ge c_1(\mathbf{A}). \tag{13}$$

With all possible changes in α and β, the vector $\mathbf{A}$, (12) describes a unit circle in a plane containing $\mathbf{A}_1$ and $\mathbf{A}_2$. From this viewpoint, the inequality (13) may be expressed in the form of the theorem [9]

The phase speed at a degeneracy point between the bottom and middle branches is not less than the greatest speed on the bottom branch on the great circle of directions described by vector $\mathbf{A}$ (12) on the sphere $\mathbf{m}^2 = 1$. (14)

At a point of triple degeneracy the vector $\mathbf{A}$ is in general directed in any arbitrary way. In this case the meaning of inequality (13) reduces to the assertion [9]

Triple degeneracy can be realized only at a point where the speed on the bottom branch is a maximum and coincides with the minimum velocity on the top branch. (15)

Incidentally, this assertion also follows directly from theorem (9).

Energy concentration

In contrast to isotropic bodies, in crystals the energy current of the wave is generally not collinear with the wave vector $\mathbf{k}$. A homogeneous distribution of wave normals $\mathbf{m}$ on the unit sphere will then often result in an orientationally very inhomogeneous distribution of the corresponding rays. The relation between these distributions is very visual on a slowness surface representing a locus of the ends of all slowness vectors $\mathbf{s} = \mathbf{m}/c(\mathbf{m})$ outgoing from the same origin. Since the wave vector $\mathbf{k}$ may be expressed as $\mathbf{k} = \omega \mathbf{s}$, the slowness surface differs from the isofrequency surface $\omega(\mathbf{k}) = const$, only by scale. Hence the two surfaces possess the same normals. The normal to the isofrequency surface is clearly parallel to the group velocity $\mathbf{v}$, since $d\omega(\mathbf{k}) = \nabla\omega(\mathbf{k})d\mathbf{k} = \mathbf{v}\cdot d\mathbf{k} = 0$ for any $d\mathbf{k}$ belonging to the surface. Thus we can state that the normal $\mathbf{n}_S$ to the slowness surface S at any point $\mathbf{s}_\alpha(\mathbf{m})$ is specified by the direction of the group velocity $\mathbf{v}_\alpha = \nabla c_\alpha(\mathbf{m})$ of the wave propagating along $\mathbf{m}$.

More directly group velocity distributions are displayed by a ray surface R, known also as a group velocity surface or a wave surface. This surface is constructed from the radius-vector $\mathbf{v}(\mathbf{m})$ quite analogously to the way the slowness surface S was constructed from the $\mathbf{s} = \mathbf{m}/c(\mathbf{m})$ radius vector. One can tell that $\mathbf{v}(\mathbf{m})$ maps the unit sphere $\mathbf{m}^2 = 1$ to the ray surface R. It is remarkable that at any point the normal $\mathbf{n}_R$ to the R surface must be parallel to the corresponding wave normal $\mathbf{m}$.

Both surfaces are shown in Fig.1 for the example of the cubic crystal *Ge*. One can see that a major part of the phase space is made up of *energy concentration* zones of the type *COC'* (Fig. 1*a*) where the group velocity $\mathbf{v}$ belongs to rather small solid angles of the type *BOB'* (Fig.1*b*) around [100] directions. Thus, the energy flux corresponding to the wave packet traveling inside some solid angle $\Delta\Omega_{\mathbf{m}}$ might be concentrated in a solid angle $\Delta\Omega_{\mathbf{v}}$ much smaller $\Delta\Omega_{\mathbf{m}}$. For some directions $\mathbf{m} = \mathbf{m}_c$ the ratio $\Delta\Omega_{\mathbf{m}}/\Delta\Omega_{\mathbf{v}}$ may become singular

$$\lim_{\Delta\Omega_{\mathbf{m}} \to 0} (\Delta\Omega_{\mathbf{m}}/\Delta\Omega_{\mathbf{v}}) = \infty . \qquad (16)$$

Such directions play a key role in the phonon focusing.

The basic quantitative characteristic of energy concentrating, originally introduced by Maris [10], relates to the left-hand side of Eqn.(16) and is known as the *enhancement factor*

$$A = d\Omega_{\mathbf{m}}/d\Omega_{\mathbf{v}} . \qquad (17)$$

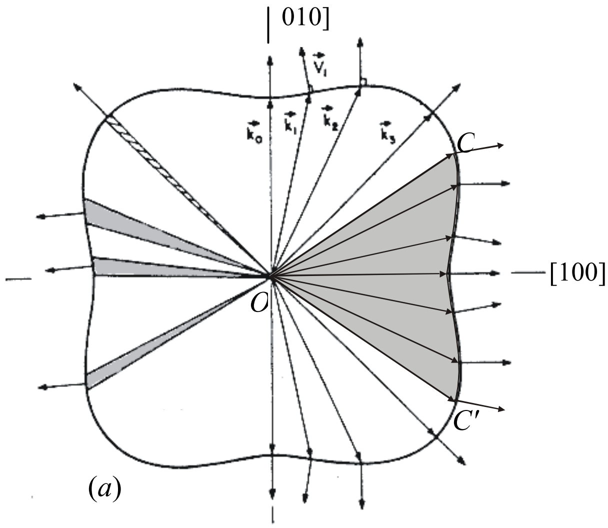
There are several approaches to the evaluation of the factor A. In particular, there is the method based on the relation between the factor A and the Gaussian curvature K of the slowness surface [11-13]:

$$A^{-1} = (v/c^3)\,|K| = c^{-2}\mathbf{n}_s \cdot \overline{\mathbf{G}}\mathbf{n}_s , \qquad (18)$$

where $\overline{\mathbf{G}}$ is the adjoint matrix to the matrix $\mathbf{G}$ with the components

$$G_{ij} = \frac{\partial^2 c}{\partial m_i \partial m_j} . \qquad (19)$$

Eqn.(18) shows that a vanishing Gaussian curvature K results in an infinite enhancement factor A (17). Solutions of the equation $K = A^{-1} = 0$ usually form closed lines on the slowness surface, known as *parabolic lines* [11, 14]. Parabolic lines are very common though not a necessary feature of crystals. As was shown by Every [14], for their existence it is sufficient for a crystal to have at least one conical acoustic axis. Parabolic lines on the slowness surface S represent in $\mathbf{k}$-space the image of the locus points relating to singularity (16) of the enhancement factor. The corresponding image in the $\mathbf{r}$-space is given by the so-called *caustics* on the ray surface R. Naturally, caustics display much more complex patterns on R than parabolic lines on *S*.



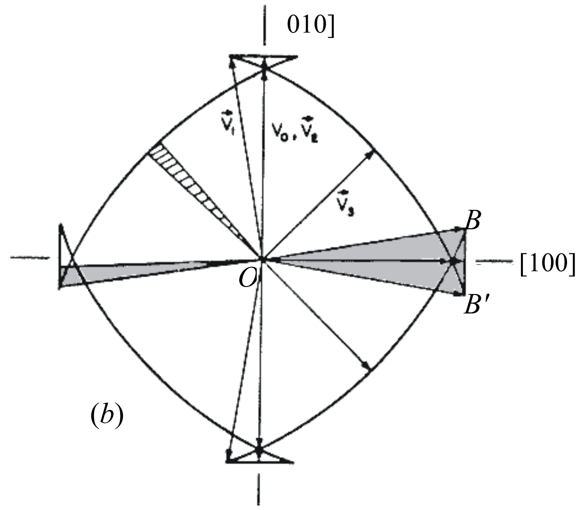


Figure 1: The example of energy concentration for Ge crystal [11]; (*a*) the outer sheet of the slowness surface; (*b*) the relating sheet of the group velocity surface

Acoustic axes in crystals

As was mentioned above, in isotropic media all directions of propagation are degenerate, $c_1 = c_2 = c_t$. In transversely isotropic materials a cone of acoustic axes may arise under the condition

$$(c_{66} - c_{44})[(c_{11} - c_{66})(c_{33} - c_{44}) - (c_{44} + c_{13})^2] > 0. \quad (20)$$

The angle θ_d of the cone around the principal axis is

$$\theta_d = \tan^{-1}\left[\frac{(c_{11} - c_{66})(c_{33} - c_{44}) - (c_{44} + c_{13})^2}{(c_{11} - c_{66})(c_{66} - c_{44})}\right]^{\frac{1}{2}} . \quad (21)$$

In any other crystals acoustic axes may arise only as separate directions. One can prove [15-17] that

In crystals of unrestricted anisotropy, however not isotropic or transversely isotropic, the total number of acoustic axes does not exceed 16. (22)

As was shown by Khatkevich [18], directions $\mathbf{m}_d$ may be determined in terms of the components Q_{ij} of the acoustical tensor. If all non-diagonal components

$$Q_{ij}(\mathbf{m}_d) \neq 0, \quad i \neq j, \qquad (23)$$

the corresponding directions $\mathbf{m}_d$ are determined by the system

$$R_1 = (Q_{11} - Q_{22})Q_{13}Q_{23} - Q_{12}(Q_{13}^2 - Q_{23}^2) = 0, \quad (24)$$

$$R_2 = (Q_{11} - Q_{33})Q_{12}Q_{23} - Q_{13}(Q_{12}^2 - Q_{23}^2) = 0. \quad (25)$$

Alshits & Lothe [19] have added to R_1 and R_2 five more components and formed the 7-component vector $\boldsymbol{\xi} = \{R_1, R_2, ..., R_7\}$. They proved that the equation

$$\boldsymbol{\xi} = 0 \qquad (26)$$

represents an invariant criterion of degeneracy valid in an arbitrary coordinate system independently of the condition (23).

Acoustic axes are very common objects in crystals. In fact, until now among practically studied materials there are no examples of crystals free of acoustic axes. And as a rule, a fastest phase-velocity branch remains nondegenerate. The only exclusion found in 1972 by Ohmachi *et. al.* [20] for TeO_2 crystals, where all three wave branches turn out to be degenerate, has prevented attempts to prove that the latter empirical observation is a general property of bulk eigenwaves in crystals. In the same way, attempts to prove a theorem of obligatory existence of acoustic axes in crystals of unrestricted anisotropy were stopped after Alshits & Lothe [9] in 1979 introduced the example of a thermodynamically stable model crystal without acoustic axes. According to [9], any orthorhombic crystal with the elastic moduli $c_{12} = c_{13} = c_{23} = 0$ and

$$0 < c_{22} < c_{66} < c_{11} < C_1 < C_2 < c_{55} << c_{33} \qquad (27)$$

must be completely free of acoustic axes. In (27) $C_{1,2}$ constants are defined by the relations

$$C_1 = \min\left\{c_{44},\ \frac{c_{55}(2c_{44} - c_{22})}{c_{55} + 2c_{44} - c_{22}}\right\},\quad C_2 = \max\left\{c_{44},\ \frac{c_{55}(2c_{44} - c_{22})}{c_{55} + 2c_{44} - c_{22}}\right\}.$$

Later an alternative example of orthorhombic medium with nondegenerate phase speed branches was also found numerically [21, 22].

It is worthwhile to mention that properties of a crystal without acoustic axes must be rather unusual. In particular, in such a medium longitudinal normals are obligatory in all three sheets of a phase velocity surface, including the “quasi-transversal” sheets. And along the latter directions in the “quasi-longitudinal” fastest sheet a purely transversal wave must propagate.

An example of a model medium free of acoustic axes would be impossible for systems of higher symmetry than orthorhombic. Any symmetry axis higher than 2-fold axis must be an acoustic axis. Accordingly, in *tetragonal crystals* only one acoustic axis along a principal 4-fold axis is obligatory, the other acoustic axes may exist or may not (altogether there could be 1, 5 or 9 acoustic axes in seven different combinations). In *trigonal crystals* there are possible only two variants: 4 (obligatory) or 10 acoustic axes. In *cubic crystals* 7 obligatory acoustic axes along 4- and 3-fold symmetry axes always exist and no other degeneracies may occur in this symmetry system. The more detailed analysis of acoustic axes in crystals of particular symmetry systems one can find in [3, 4, 15, 17, 22, 23].

Though orientations of acoustic axes are determined by the same equation (26) and the basic characteristics of the eigenwaves propagating along $\mathbf{m}_d$ are universal, degeneracy directions differ from each other by their neighborhood. They can be classified by geometrical types of contact along $\mathbf{m}_d$ of degenerate velocity sheets or / and by types of singularities of vector polarization fields $\mathbf{A}_{\alpha,\beta}(\mathbf{m})$ around $\mathbf{m}_d$.

Geometrically one should distinguish conical and tangent points of contact and lines of intersection of degeneracy sheets (the latter for transverse isotropy, see Eqn.(21)). For model crystals there are known also the so-called wedge-point contacts and lines of tangency. Alshits & Lothe [19, 24] (see also [25, 26]) first noticed that geometrical features of degeneracies correlate with definite types of polarization singularities. This observation was elaborated in [27] where a complete classification of acoustic axes was constructed including all possible types of local geometry of the velocity sheets near the degeneracy, and the corresponding polarization singularities. The developed theory also provides algebraic conditions for any type of degeneracy, without solving the wave equation for arbitrary anisotropy, but using only appropriate convolutions of the tensor c_{ijkl}.

As an example, let us construct the two vectors

$$\mathbf{p} = (\mathbf{S}_{11} - \mathbf{S}_{22})\mathbf{m}_d, \qquad \mathbf{q} = (\mathbf{S}_{12} + \mathbf{S}_{21})\mathbf{m}_d, \quad (28)$$

where $\mathbf{S}_{\alpha\beta} = (A_\alpha A_\beta)/2\rho c_d$ with c_d and $\mathbf{A}_{1,2}$ (19) being the degenerate phase speed and the orthogonal pair of unit vectors in the degeneracy plane, Eqn.(12). It turns out that the vectors $\mathbf{p}$ and $\mathbf{q}$ determine the geometry of the contact of the velocity sheets in the degeneracy point $\mathbf{m}_d$. If the vector product $\mathbf{p} \times \mathbf{q}$ does not vanish, one has a degeneracy of the *conical type*. It is customary to characterize point singularities in plane distributions of vector fields by a *topological “charge”* - a Poincaré index n, which is defined as the angle (in 2π units) of aggregate rotation of vectors around a singular point. In these terms the topological charge of the conical degeneracy at $\mathbf{m}_d$ is

$$n = \tfrac{1}{2}\mathrm{sgn}[\mathbf{m}_d(\mathbf{p} \times \mathbf{q})]. \qquad (29)$$

If $\mathbf{p} \times \mathbf{q} = 0$ but $\mathbf{p}$ or $\mathbf{q}$ does not vanish, we arrive at a *wedge degeneracy* at the point $\mathbf{m}_d$ or along the line passing through $\mathbf{m}_d$. The case $\mathbf{p} = \mathbf{q} = 0$ corresponds to a degeneracy of the *tangent type* at a point or along a line. In order to distinguish the degeneracy at a point from that along a line and in order to calculate the Poincaré index for a wedge-point degeneracy

($n = 0, \pm\frac{1}{2}$) or for a tangent-point degeneracy ($n = 0, \pm 1$), one should use other convolutions of the tensor c_{ijkl} with the vectors $\mathbf{m}_d$ and $\mathbf{A}_\alpha$, see [27, 28].

Almost all types of acoustic axes exist in real crystals. Conical degeneracies ($n = \pm\frac{1}{2}$) exist in all known crystals of orthorhombic, monoclinic and triclinic symmetry systems. Along a 3-fold axis both in trigonal and in cubic crystals the conical degeneracy with the Poincaré index $n = -\frac{1}{2}$ always occurs. A 4-fold axis in tetragonal and cubic crystals is always a tangent acoustic axes with the Poincaré index $n = \pm 1$. For the cubic system the choice of the sign of n is especially simple: $n = \mathrm{sgn}(c_{12} + c_{44})$. An ∞-fold axis in transverse isotropic crystals is also always a tangent acoustic axis, however its topological charge is definite: $n = 1$. In crystals of this symmetry one can also meet the wedge-line degeneracy, Eqns.(20), (21). In the model crystal, where apart from (20) also the condition $c_{44} = c_{66}$ is satisfied, the two symmetrical wedge lines in accordance with (21) must coalesce into one tangent degeneracy line in the basal plane ($\theta_d = \pi/2$).

Different types of acoustic axes manifest a different behavior (disappearance, shifting, splitting) under small perturbations of the elastic properties. The analysis of this problem in [19, 27] reveals that only acoustic axes of the conical type are always stable under perturbations, i.e. they can only shift. Unstable points of degeneracy either split in accordance with the rule of conservation of topological charge or vanish but only if $n = 0$. For instance, at a phase transition from a transverse isotropic to a trigonal crystal the ∞-fold axis ($n = 1$) is replaced by the 3-fold axis ($n = -\frac{1}{2}$). In accordance with the rule of index conservation and with symmetry requrements in addition to the latter degeneracy there must arise also three conical acoustic axes of the index $n = \frac{1}{2}$ (Fig.2).

As was mentioned in [27] and studied in [29], even a conical degeneracy is unstable with respect to the "switching on" a small damping, which is equivalent to a small imaginary perturbation of the phase speed, $v \to v - i\delta v$. As a result of such perturbation, the conical axis split into a pair of singular directions connected on the slowness surface and on the surface of damping by lines of intersection of corresponding sheets. The only two common inversely nonequivalent points of these lines on the unit sphere $\mathbf{m}^2 = 1$ correspond to new positions of acoustic axes. The polarizations of degenerate waves along the two new singular directions are circular. In the vicinity of these points of degeneracy, polarizations are elliptic. The rotation of large semi-axes of these ellipses around each of degeneracy points corresponds to the Poincaré index $n = \pm\frac{1}{4}$, which fits the index conservation rule.

The extension of the theory [27] to piezoelectrics was given in [30, 31]. It was found that the classification of acoustic axes does not change apart from renormalization of explicit forms of the algebraic conditions for particular types of degeneracies, due to contributions from piezoelectric moduli. However this contribution may qualitatively change the wave properties near a specific acoustic axis and even change the type of the degeneracy itself, see [31, 32]. Piezoelectric coupling also causes the quasi-static electric field accompanying the elastic waves. Its characteristics depend on the polarization of the elastic wave and therefore also have definite singularities near the directions $\mathbf{m}_d$ [30].

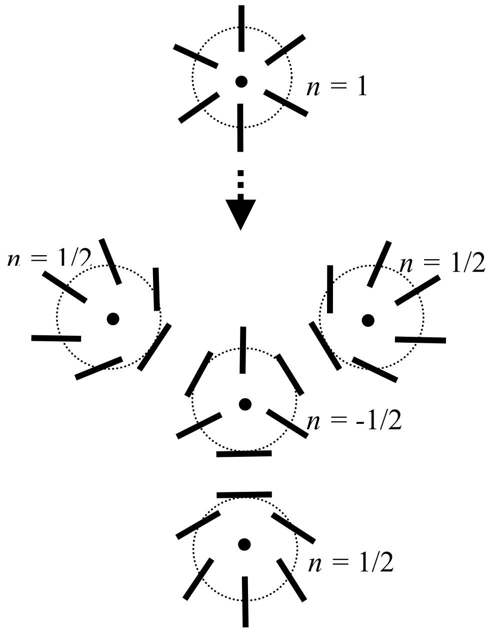


Figure 2: Splitting of the tangent degeneracy along the ∞-fold axis into the four conical degeneracies at the phase transition from the hexagonal to the trigonal symmetry system

Surface, leaky, exceptional and quasi-bulk waves

Stroh sextic formalism

Consider a half-infinite medium of unrestricted anisotropy. Let us choose the coordinate system with the origin at the surface, the y-axis along the internal normal $\mathbf{n}$ to the surface and the x-axis along the direction $\mathbf{m}$ of wave propagation. The plane specified by the unit vectors $\mathbf{m}$ and $\mathbf{n}$ is known as a *sagittal plane*. In this coordinate system the steady-state displacement field of the plane wave can be presented as a superposition of partial waves,

$$\mathbf{u}(x - vt, y) = \sum_\alpha b_\alpha \mathbf{A}_\alpha \exp[ik(x + p_\alpha y - vt)], \quad (30)$$

which have equal x-components of the wave vector, $k_x \equiv k$, and a common tracing speed $v = \omega / k$, but different y-components of the wave vector, $k_y^{(\alpha)} = p_\alpha k$, and polarization vectors $\mathbf{A}_\alpha$. Stroh [33] proposed, in parallel with (30), to consider the *traction field*,

$$\mathbf{n\sigma}(x - vt, y) = -ik\sum_\alpha b_\alpha \mathbf{L}_\alpha \exp[ik(x + p_\alpha y - vt)], \quad (31)$$

and proved that the 6-vectors $\xi_\alpha = (\mathbf{A}_\alpha, \mathbf{L}_\alpha)^T$ and the parameters p_α are eigenvectors and eigenvalues of some real 6×6 matrix $\mathbf{N}$: $\mathbf{N}\xi_\alpha = p_\alpha \xi_\alpha$. This equation leads to a secular equation with real coefficients, determining six functions $p_\alpha(v)$, which must be real or form complex conjugate pairs. Here and below, excluding *transsonic states*, we suppose all p_α to be non-degenerate. The first transsonic state takes place at the so-called *limiting velocity* $\hat{v}$, which is defined as the lowest tracing speed admitting a bulk mode in (30). The range $0 < v < \hat{v}$ is called a *subsonic range*. Here all eigenvalues p_α and eigenvectors ξ_α occur in pairs of complex conjugates. At $v = \hat{v}$ one of the conjugate pairs coalesces into one degenerate eigenvalue $\hat{p}$, which in the *supersonic range* $v > \hat{v}$ splits into a pair of real parameters. On the other hand, in the range $v < \hat{v}$ three of six inhomogeneous terms of superposition (30) contain infinitely increasing exponents at $y \to \infty$ and the corresponding three amplitudes b_α must vanish. With the choice of numeration providing $\mathrm{Im}\, p_\alpha > 0$ for α = 1, 2, 3, there should be $b_{4,5,6} = 0$. As a result, the solution (30)-(31) describes a 3-partial field localized at the surface, i.e. a surface wave. Its amplitudes b_α are supposed to be found from the boundary conditions. For a free surface the requirement of vanishing traction $\mathbf{n\sigma} = 0$ reduces to the equation $b_1\mathbf{L}_1 + b_2\mathbf{L}_2 + b_3\mathbf{L}_3 = 0$, which may have non-trivial solutions only if $\det\{L_{\alpha i}\} = 0$. This equation determining the velocity v_R of the Rayleigh wave is complex. For this reason J.L. Synge (1956) supposed that this boundary problem is over-determined and the forbidden directions for surface wave propagation in anisotropic bodies were likely to be the rule rather than the exception (see [34]). However, Stroh [33] has proved that $\det\{L_{\alpha i}\}$ has a structure $(a + ib) f(v)$, where $f(v)$ is a real function. Thus the dispersion equation is real and surface waves in anisotropic media are not forbidden.

Existence considerations

Of course, the reality of the equation $f(v)$ = 0 does not guarantee an existence of its solutions. In 1973 Barnett *et al.* [35] proved the *uniqueness theorem* of a solution for a surface wave, when it exists in the region $v < \hat{v}$. Later Barnett & Lothe [36-38] and Chadwick & Smith [39] proved also the *existence theorem* for subsonic surface waves, gradually sharpening its formulation on the basis of establishing new and new mathematical properties of different quantities involved into the theory.

Before presenting the final formulation of this theorem let us return to the concept of a first transonic state $v = \hat{v}$, which plays a key role in the existence conditions. Usually at $v = \hat{v}$ there is only one point of tangency between the vertical line and the outer sheet of the slowness surface. Such configuration is called Type 1 transonic state. The corresponding bulk limiting wave would propagate with the group velocity parallel to the surface, however this might happen only in the exceptional situation when this wave satisfies the condition of a free surface. Such waves, known as *exceptional bulk waves* [39], occupy only 1-dimensional sub-space (lines) in the 3D space of all possible surface wave geometries [26], i.e. orientations of the frame $\{\mathbf{m}, \mathbf{n}\}$. The other five types of transonic state arise [39] when the vertical line $v^{-1} = \hat{v}^{-1}$ is tangent simultaneously with more than one sheet of the slowness surface at the same point or at different points of the same sheet.

So, the existence theorem may be stated in the form:

The existence of a surface wave at a free surface of an elastic half-space is guaranteed in the subsonic range $v < \hat{v}$ if the first transonic state is not of Type 1, or if it is of Type 1 but the corresponding limiting wave is not exceptional. (32)

If the first transonic state is of Type 1 and the limiting wave at $v = \hat{v}$ is exceptional, then a surface wave solution in the range $v \le \hat{v}$ may exist or may not. Also the *non-existence theorem* is valid [36-38]:

A clamped surface cannot support a free localized wave in the elastic half-space. (33)

Of course, the conditions of a free or clamped surface do not represent a complete list of physically possible boundary problems for surfaces waves. The alternative problem of a loaded boundary, $\mathbf{\sigma n} = -\hat{\lambda}\mathbf{u}$, has been considered by Alshits *et al.* [40] for a scalar or tensor coefficient $\hat{\lambda}$. It was found that, depending on the parameters of the loading system, the boundary problem may admit several solutions or none.

The Stroh approach was later extended [41-43] so that it could be applied to piezoelectric crystals. This has led to an 8-dimensional formalism, which allowed extending theorems (32), (33). The non-existence theorem for a clamped surface remains valid also for a piezoelectric medium. However, a uniqueness property generally does not retain. In piezoelectrics there are possible two surface waves solutions (e.g. of Rayleigh or Bleustein-Gulyaev type). But only one of them is guaranteed for a free metallized surface, when the limiting wave is not exceptional. Otherwise, two, one or none solutions may occur in this case.

There are also crystal classes where simultaneously piezoelectric, piezomagnetic and magnetoelectric couplings coexist. An analysis of the surface wave existence in such media was accomplished [44] in the framework of the 10-dimensional formalism. Again two, one or none solutions are possible. Only one wave is guaranteed when the limiting wave is not exceptional and the surface is mechanically free and both electrically and magnetically closed. It is noteworthy that here we meet for the first time the case when a *clamped surface* may admit the existence of a surface wave solution. In [45] such solutions were explicitly found for a model thermodynamically stable crystal. The other nontrivial situation was indicated in [46], where a new branch of surface magneto-electro-elastic waves, existing completely due to magneto-electric coupling, was theoretically predicted in a piezomagnetic material adjoining a superconducting or a superdiamagnetic material without mechanical contact. When the magnetoelectric interaction is "switched off", these surface waves disappear or transform into bulk waves.

We note that specific surface waves in crystals with piezoelectric, piezomagnetic or magnetoelectric couplings arise due to anisotropy and do not exist in isotropic bodies. On the other hand, new surface modes are known even in purely elastic anisotropic media. We can mention *quasi-bulk surface waves* [47], which may arise at a perturbation of the propagation geometry of bulk exceptional waves, *supersonic surface waves* with a speed $v > \hat{v}$ [48], which form lines of solutions [49] in the 3D space of propagation orientations and *leaky waves* [48, 50].

Conclusion

Thus, as we have seen, anisotropy in acoustics of crystals is not at all reduced just to more complex theoretical descriptions of known wave properties. It creates principally new kinds of waves and gives rise to new acoustic phenomena. On the other hand, as we tried to demonstrate, anisotropy in crystalloacoustics is a beautiful world of elegant mathematics, non-trivial theorems, unexpected peculiarities, singularities and topological catastrophes. In our considerations we have limited ourselves to the analysis of acoustic wave properties only in infinite and half-infinite anisotropic media. In fact, the same methods have proved to be very efficient in many other acoustic problems. We can mention the theory of interfacial waves in anisotropic bicrystalline structures, the theory of fundamental and wave-guided eigenmodes in anisotropic plates, and the studies of acoustic spectra in anisotropic layer-substrate structures.

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