RADIATIVE STRESSES OF ACOUSTIC WAVES IN SOLIDS

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Abstract

The physical meaning of the notions of wave momentum and radiative stress in elastic bodies is considered. Of great importance for defining and calculating these quantities are the variables in which the wave processes in the medium are described. In Euler variables, the radiative stress is defined as the time average of the momentum flux of the medium across the boundary of a volume that is stationary in space. It is shown that, physically, the wave momentum can be defined only in Euler variables. In the linear approximation of the small perturbation theory the wave momentum is a quadratic component of the total mechanical momentum of the medium. In the second approximation of the perturbation theory, the wave momentum no longer has a rigorous definition. However, the concept of wave momentum borrowed from the linear approximation can be used for the calculation of average values of stresses and deformations, provided the body is not moving as a whole.

Introduction.

When wave processes are investigated in elastic bodies, in addition to the variables such as displacement, oscillatory velocity and so on, quantities that are constant in time, i.e., are *conserved*, are encountered too. These include quantities that are quadratic in the wave amplitude, in particular, energy and wave pressure (radiative stress). The origin of radiative stresses is related to the change of the time averaged momentum of the medium transported by the wave [1-3]. The change of wave momentum may be caused by scattering by inhomogeneities, reflection on obstacles, absorption, or radiation. The notions of wave momentum and radiative stresses have been employed in the analysis of wave processes in continuous media but not so often, so that the present study aims at revealing the physical meaning and the contents of these notions for wave processes in elastic media.

The radiative stress (wave pressure) and the related wave momentum are typical for waves of arbitrary nature: electromagnetic, optical, acoustic, surface waves on water, and elastic waves in solids. However, while the wave pressure and momentum have been investigated in much detail, both theoretically and experimentally, for optical and electromagnetic waves in a vacuum, the existence and physical meaning of the wave momentum in a continuous medium are still not quite transparent and are still being discussed in the scientific literature, so that there is need for a precise study. The definition of wave momentum and radiative stresses for waves in elastic bodies differs from that for electromagnetic waves or acoustic waves in liquid and gaseous media.

The wave pressure and momentum which we are interested in are quadratic in wave amplitudes and, consequently, are quantities of the second order of smallness as compared to the instantaneous elastic stresses and deformations. In a general case, solution of the problem in a linear approximation is insufficient for calculation of wave momentum and pressure in a medium, and quantities in the second approximation need to be taken into account. For the explanation of the phenomenon of wave momentum in problems of hydro- and gas dynamics, nonlinear equations of the medium motion need to be analysed. In this case, appearance of wave momentum is attributed to the secondary flows generated by the acoustic field, and one has to go beyond the scope of one-dimensional problems and consider sound beams and secondary flows produced by the acoustic field. In contrast to hydrodynamics, there are no secondary flows in elastic bodies, and the wave momentum and radiative stresses can be addressed in the framework of a one-dimensional problem, with allowance for the nonlinearity of the equations of motion.

The motion of a medium in continuum mechanics is traditionally described by two methods: in Euler variables related to the reference frame of an observer (laboratory frame of reference), and in Lagrangian variables strictly attached to particles of the medium at the initial instant of time. For the correct physical interpretation of theoretical results it is essential to know in which variables kinematic and dynamical characteristics of the medium motion are calculated.

In Euler variables these quantities are calculated for the medium enclosed in a definite volume of space that is stationary relative to the observer. In Lagrangian variables these quantities are calculated for a volume of the medium consisting of the same particles. Consequently, this volume is constantly changing relative to the observer, with the law of motion of a body, the bounding surface being determined by the motion of particles in the medium. Hence, it should be born in mind that, from the viewpoint of the observer, kinematic and dynamical quantities in Euler and Lagrangian variables are calculated for different physical situations and, generally, lead to different results. Calculations show that such discrepancies are manifested in the second and higher orders of the perturbation theory only. But the quantities of radiative stress and wave momentum of interest to us are of the second order in wave amplitudes; therefore the difference between the Euler and Lagrangian descriptions of wave motions need to be taken into account [5].

General relationships.

We suppose that no external forces act on the medium where the motions occur only due to internal stresses. In Euler variables (x_k, t), the isentropic (adiabatic) deformations of the elastic medium are governed by equations:

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} \qquad (1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}\rho v_k = 0 \qquad (2)$$

$$dU = \frac{1}{\rho}\sigma_{ik} d\varepsilon_{ik} \qquad (3)$$

Here, v_i is the velocity of particles in the medium, ρ is the density, σ_{ik} and ε_{ik} are the stress and strain tensors, respectively, and U is the internal energy per mass unit. This set of equations is closed by adding expressions relating the medium deformation and the velocity of its particles to the displacement field $u_i(x_k, t)$:

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right), \qquad v_i = \frac{\partial u_i}{\partial t} + v_k \frac{\partial u_i}{\partial x_k} \qquad (4)$$

From equations (1)-(3) we can obtain equations for the transfer of momentum $q_i = \rho v_i$ as

$$\frac{\partial q_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0 \ , \qquad (5)$$

where the $\Pi_{ik} = (q_i v_k - \sigma_{ik})$ is the momentum flux density that has the dimension of stress. From (5) it follows that the change of momentum in a finite volume $Q_i = \int_V q_i dV$ is equal to [1, 2]

$$\frac{dQ_i}{dt} = -\oint_S \Pi_{ik} n_k ds \qquad (6)$$

For a the sake of simplicity, in further considerations we shall consider only a one-dimensional problem concerning travelling longitudinal waves in a thin rod. Right subscripts x and t will now denote time and space derivatives. In this case the time rate of change of momentum for a segment of a rod looks like:

$$\frac{dQ}{dt} = -\Big[\rho v^2 - \sigma\Big]_{x_1}^{x_2} = -\big[\Pi(x,t)\big]_{x_1}^{x_2} \qquad (7)$$

Here, the square brackets denote the difference. Equation (7) is a generalization of the second Newton law to a finite element of the continuous medium confined between fixed sections x_1 and x_2. Consequently, the right-hand side of (7) contains an expression for the a pseudo-force acting on a medium element $\Delta x = x_2 - x_1$:

$$F = -\Big[\rho v^2 - \sigma\Big]_{x_1}^{x_2} = -\big[\Pi(x,t)\big]_{x_1}^{x_2} \qquad (8)$$

Following L. Brillouin [3], the time independent component of force F :

$$P = \langle F \rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T F dt = \Big[\big\langle \sigma - \rho v^2 \big\rangle\Big]_{x_1}^{x_2}, \qquad (9)$$

where T is the time interval of averaging, will be called *radiative stress* (i.e., the "force of wave pressure"). Hereinafter we suppose that parameters ρ, σ and v change only under the action of a deformation wave. From the definition of the *radiative stress* (9) it follows that it differs from the constant component of elastic stress $\langle \sigma \rangle$ by a value equal to twice the kinetic energy. The portion of the total momentum of the medium, ρv, related to the *deformation wave* will be referred to as *wave momentum*.

Note that the wave momentum is not identical to the total momentum of the medium because the wave energy is not equal to the total energy of the medium. In some works, to avoid confusion between total momentum of the medium ρv and wave momentum, the latter is called pseudomomentum. The expression of the wave momentum starts with quadratic quantities in the wave amplitude. Therefore its time average for a travelling periodic wave is nonzero. This indicates that a longitudinal wave travelling in one direction cannot be excited in a rod if a nonzero momentum is not specified in the medium.

Our further analysis will be carried out using the method of successive approximations assuming that in the neighbourhood of the stationary state the sought quantities can be represented in the form

$$\begin{aligned} &\rho = \rho_0 + \rho' + \rho'', \\ &u = u' + u'', \ \sigma = \sigma' + \sigma'', \\ &\nu = \nu' + \nu'' = u'_t + u'_x u'_t + u''_t, \\ &\varepsilon = \varepsilon' + \varepsilon'' = u'_x + 1/2(u'_x)^2 + u''_x \end{aligned} \tag{10}$$

where one prime denotes quantities of the first order of smallness $(\rho'/\rho_0 = \mu << 1)$, and two primes are for quantities of the second order of smallness $(\rho''/\rho' \sim v''/v' \sim \mu << 1)$. The expressions expanded in terms of perturbations for the momentum density to an accuracy of $O(u_x'^3)$ are written in the form

$$\rho v = \rho_0 v' + \rho' v' + \rho_0 v'' + O(\rho' v''). \tag{11}$$

The force of wave pressure on an absolutely absorbing obstacle (i.e., $\Pi(x_2) = 0$), to the same accuracy, is equal to

$$P = -\rho_0 \langle v'^2 \rangle + \langle \sigma' \rangle + \langle \sigma'' \rangle + O(\langle \rho' v'^2 \rangle). \tag{12}$$

For nonlinear longitudinal waves in a thin rod stress and strain are coupled by a relation $\sigma = E\varepsilon + \alpha \varepsilon^2$, where E and α are the linear and nonlinear elasticity coefficients, respectively

The effects of the first approximation

In this case the small perturbations in a rod caused by a deformation wave will be described by a linear equation for displacement $u'(x,t)$:

$$\rho_0 \frac{\partial^2 u'}{\partial t^2} - E \frac{\partial^2 u'}{\partial x^2} = 0, \tag{13}$$

and perturbations of all the other quantities will be expressed through its derivatives

$$\begin{aligned} &\rho' = -\rho_0 u'_x, \quad v' = u'_t, \\ &\varepsilon' = u'_x, \quad \sigma' = Eu' \end{aligned} \tag{14}$$

It is easy to show that the wave equation has the integral of motion (strict conservation law)

$$\frac{\partial}{\partial t}(-\rho_0 u'_t u'_x) + \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\rho_0 u_t'^2 + E u_x'^2\right)\right] = 0, \tag{15}$$

that is obtained from (13) by multiplying it by u'_x, rearranging and grouping. This equation represents the transport of *wave momentum*

$$g = -\rho_0 u'_t u'_x. \tag{16}$$

The quantity numerically equal to the energy of a linear wave is in this case the density flux of wave momentum (Note: This may be an artifact of the one-dimensional nature of the discussed problem; cf. Maugin [4], p.211). The equation for the transport of momentum (5) is then written in the form

$$\frac{\partial}{\partial t}(\rho_0 u'_t - \rho_0 u'_t u'_x) + \frac{\partial}{\partial x}(\rho_0 u_t'^2 - E u'_x) = 0. \tag{17}$$

One can readily see that the linear terms satisfy the equation of motion (13), and the quadratic terms give

$$\frac{\partial}{\partial t}(-\rho_0 u'_t u'_x) + \frac{\partial}{\partial x}(\rho_0 u_t'^2) = 0 \tag{18}$$

that coincides (*on the average in time*) with the equation for the transfer of wave momentum (15) because $\langle \rho_0 u_t'^2 \rangle = \langle (1/2)(\rho_0 u_t'^2 + E u_x'^2) \rangle$. In this approximation, the momentum density (11) per unit length is made of two contributions since

$$q = \rho_0 u'_t - \rho_0 u'_t u'_x \tag{19}$$

The first contribution is a linear function of the particle velocity. It is always related to mass transport. It differs from zero even in the absence of deformation wave, e.g., during uniform motion of a rod as a whole when $u'_x = 0$ and $u'_t \neq 0$. The second contribution to the momentum density, is a quadratic quantity with respect to the wave variable $u(x,t)$; it differs from the first contribution significantly. It is generated due to the presence of a *deformation wave* in the rod and it reduces to zero in its absence. The time average of the difference of its fluxes across sections x_1 and x_2 determines the wave pressure force acting on the element of the rod in the direction of wave propagation.

The effects of the second approximation

The equation of momentum transfer takes the form

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho_0 u'_t - \rho_0 u''_t) = \\ &= \frac{\partial}{\partial x}\left[-E u''_x + \rho_0 u_t'^2 - E u'_x - \left(\alpha - \frac{E}{2}\right) u'_x\right]. \end{aligned} \tag{20}$$

Taking into account that quantities in the first approximation satisfy equation (13) we obtain a linear inhomogeneous equation for quantities of the second order of smallness

$$\rho_0 \frac{\partial^2 u''}{\partial t^2} - E \frac{\partial^2 u''}{\partial x^2} = \\ = \frac{\partial}{\partial x}\left[\left(\alpha - \frac{E}{2}\right)\left(\frac{\partial u'}{\partial x}\right)^2 - \rho_0\left(\frac{\partial u'}{\partial t}\right)^2\right] \quad (21)$$

All the other second-order perturbations ρ'', v'', ε'', σ'' are expressed through derivatives of u' and u'' by the following formulas

$$\rho'' = -\rho_0 u''_x,$$
$$v'' = u''_t + u'_x u'_t,$$
$$\varepsilon'' = u''_x - \tfrac{1}{2}u'^2_x, \quad (22)$$
$$\sigma'' = Eu''_{xx} + \left(\alpha - \frac{E}{2}\right)u'^2_x$$

For obtaining a closed problem in the second approximation, equation (21) must be supplemented by initial and boundary conditions. One should bear in mind that boundary conditions should be introduced already for variable values of coordinates because displacements of the boundaries also have the second order of smallness. Equation (21) is an equation for the linear wave $u''(x,t)$ in a medium with distributed source of external force in the right-hand side, the density of which is determined by the first-order perturbation $u'(x,t)$. The expressions for the momentum density (11) and wave pressure force (12) in the second approximation are written as

$$q = \rho_0 u'_t + \rho_0 u''_t \quad (23)$$

$$P = \Big[-\rho_0\langle u'^2_t\rangle + E\langle u'_x\rangle + \Big]_{x=x_1}^{x=x_2} + .$$
$$+\left[\left(\alpha - \frac{E}{2}\right)\langle u'^2_x\rangle + E\langle u''_x\rangle\right]_{x=x_1}^{x=x_2} \quad (24)$$

Note that, whereas new quadratic terms enter the expression for P only additively, the expression for momentum density (23) is significantly altered in comparison with the first approximation (19).

The solution of the problem in the second approximation is much more complex because in Euler variables the second boundary condition must be written at a *movable boundary* $x = l(t)$. However, some features of wave processes in the second approximation that are of interest to us can be found without solving the corresponding boundary value problem [5].

Conclusions

In defining the physical meaning of one or another theoretical result in continuum mechanics one should proceed from the fact that an observer (experimentalist) is in a laboratory frame of reference that consists of a time and spatial (Euler) coordinate system. Therefore, the quantities calculated in Euler variables are regarded to be observable (i.e., true) physical quantities. Thus, the wave pressure force and the wave momentum defined in Euler coordinates should be considered as physically meaningful variables. If one consistently keeps to these definitions of radiative stress and wave transported momentum, then the same physical results will be obtained, independently of the method of description of the wave field.

A sufficiently rigorous mathematical definition of the notion of wave momentum is possible only at the first approximation in Euler variables. It is a quadratic component of the body momentum related to variations of the medium density in the wave field (16). Wave momentum no longer has a rigorous mathematical definition in the second approximation. But its notion can be successfully used in a number of problems that require calculation of averaged values, for instance, when there is no time averaged body motion.

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