PMLs for the numerical simulation of harmonic diffracted waves in an elastic plate

E. Bécache[†,*], **A. S. Bonnet-Ben Dhia**[‡,*]
[†]INRIA ROCQUENCOURT, Projet Ondes, UMR 27-06 CNRS-INRIA-ENSTA, France
[‡]ENSTA, SMP, UMR 27-06 CNRS-INRIA-ENSTA, France
[*]Email: eliane.becache@inria.fr, bonnet@ensta.fr

Abstract

This work concerns the numerical finite element computation, in the frequency domain, of the diffracted wave produced by a defect (crack, inclusion, perturbation of the boundaries etc..) located in an infinite elastic plate. The Perfectly Matched Layers are used to bound artificially the computational domain. An original formulation is considered and implemented, whose unknowns are the total field in the physical domain and the diffracted one in the absorbing layers. For a homogeneous isotropic plate, the outgoing solution is well selected, except for frequencies where some propagative modes have group and phase velocities of opposite signs.

Introduction

We consider a 2D homogeneous isotropic elastic plate (of thickness $2h$, density ρ and Lamé's coefficients λ and μ) with a local perturbation of its free boundaries. The perturbed (respectively unperturbed) domain is denoted by Ω (resp. Ω_0). Our purpose is the computation of the wave diffracted by the perturbation, when the incident wave is supposed to be a propagative Lamb mode (see for instance [12]):

$$u_{inc}(x,y) = v(y)e^{i\beta x}, \quad (x,y) \in \Omega_0, \quad \beta \in \mathbb{R}.$$

If $\omega > 0$ denotes the pulsation (the term $e^{-i\omega t}$ will be omitted in the following), the total displacement field u_{tot} then satisfies :

$$\begin{cases} -\operatorname{div}\sigma(u_{tot}) - \omega^2 \rho u_{tot} = 0 & \text{in } \Omega, \\ \sigma(u_{tot}).n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\sigma(u)$, the stress tensor, is related to the strain tensor $\varepsilon(u) = 1/2(\nabla u + \nabla^t u)$ by Hooke's law:

$$\sigma(u) = \lambda \operatorname{div}(u) Id + 2\mu\varepsilon(u).$$

The diffracted wave is then defined in $\Omega \cap \Omega_0$ as

$$u_{dif} = u_{tot} - u_{inc}$$

and has to satisfy an outgoing radiation condition. An explicit expression of this condition can be derived from the expansion of u_{dif} on Lamb modes (see [7] for a mathematical justification of this expansion). More precisely, on each side of the perturbation, this expansion must involve only "outgoing" modes:

- an evanescent mode ($\beta \notin \mathbb{R}$) is said to be outgoing if $\Im m\beta > 0$ for $x \to +\infty$ and $\Im m\beta < 0$ for $x \to -\infty$
- a propagative mode ($\beta \in \mathbb{R}$) is said to be outgoing if its group velocity $\partial\omega/\partial\beta$ is positive for $x \to +\infty$ and negative for $x \to -\infty$

In the following we will denote by $u_n^+(x,y) = e^{i\beta_n x}v_n^+(y)$ (resp. $u_n^-(x,y) = e^{-i\beta_n x}v_n^-(y)$) the rightgoing modes (resp. leftgoing modes).

In order to solve the problem with finite elements, one has to bound the computational domain. For similar problems of scalar type, exact transparent boundary conditions can be set on vertical artificial boundaries, these conditions, so-called Dirichlet-to-Neumann or impedance conditions, being derived from the modal decomposition of the diffracted field ([1]). This technique, whose only drawback is to be non-local (leading to partially full matrices), is unfortunately difficult to extend to vectorial problems, as this one. Actually, the well-suited operator to ensure the good properties (completeness, orthogonality) of Lamb modes is not the operator that relates the displacement to the normal stresses (cf [9]). This is what has motivated us to develop an alternative method, based on the use of perfectly matched layers.

Perfectly matched layers

Perfectly Matched Layers (PMLs) were introduced by Bérenger [3] in order to design efficient numerical absorbing boundary conditions (more precisely absorbing layers) for the computation of time-dependent solutions of Maxwell's equations in unbounded domains. They have since been used for numerous applications, mostly in the time domain [4], [10] but also for time-harmonic wave-like equations [13], [6], [1].

The purpose of the method is to provide a fictitious, absorbing medium, such that its interface with the "physical" bounded domain does not produce any reflection. Transposing Bérenger's formulation in the frequency domain from its original setting in the time domain consists in making the following substitution in the layers:

$$\frac{\partial}{\partial x} \longrightarrow \alpha \frac{\partial}{\partial x}, \tag{2}$$

where α is a complex number [4], [11], [5]. The operators div and σ then become $\operatorname{div}_\alpha$ and σ_α in the layers. On the interfaces between the physical domain and the absorbing layers, the solution must satisfy the continuity of the displacement and the continuity of the generalized normal stresses, defined on the left (resp. right) layer side as $\sigma_\alpha(u)n_\alpha$ where $n_\alpha = (\alpha, 0)^t$ (resp. $(-\alpha, 0)^t$). For any α, it can be easily shown that the interface between the PML and the physical domain is perfectly transparent. Moreover, it is essential for the numerical purpose that transmitted waves decrease exponentially in the layers.

It is easy to check that the modes in the absorbing layers are simply derived from the Lamb's ones as:

$$u^{\pm}_{n,\alpha}(x, y) = e^{\pm i\beta_{n,\alpha}x}v^{\pm}_n(y),\ n \in \mathbb{N}, \qquad (3)$$

with

$$\beta_{n,\alpha} = \frac{\beta_n}{\alpha}.$$

The PML model corresponds to an absorbing layer if it transforms all outgoing modes, especially the propagative ones, into evanescent modes. This leads us to look for values of α such that $\Im m\beta_{n,\alpha} > 0$, $\forall n \in \mathbb{N}$. Clearly the transformation

$$S_\alpha\ :\ \mathbb{C} \to \mathbb{C}\,, \quad z \mapsto \frac{z}{\alpha}$$

due to the change of variable used in the PML is a similarity of ratio $1/|\alpha|$ and angle $\arg\,(1/\alpha)$ around the origin in the complex plane. In most situations, propagative modes have phase and group velocities of the same sign, so that for these modes $\beta_n > 0$. In these cases, the following requirements on α:

$$\mathrm{Re}(\alpha) > 0,\ \mathrm{Im}(\alpha) < 0, \qquad (4)$$

ensure that all outgoing propagative modes become evanescent in the layers. Unfortunately, for some frequencies there exist "inverse modes" which have phase and group velocities of opposit sign, and thus become unstable in the layers. In the time domain, these modes have been proved to be responsible for the existence of numerical instabilities [2]. For our problem which is set in the frequency domain, inverse modes will not provide instabilities but they will lead to a bad selection of the outgoing wave. This situation is therefore different from the one studied in [1] where PMLs work even in the presence of inverse modes.

Variational formulation

In practice, one has to bound the computational domain and layers are of finite length L.
We denote by: Ω^L the truncated computational domain, $\Omega^L_\pm$ the layers, $\Sigma_\pm$ the internal boundaries of the layers, $\Sigma^L_\pm$ the external boundaries of the layers and Ω_b the domain contained between the two interfaces $\Sigma_\pm$ (see figure 1). The PMLs are most often presented to solve diffraction problems with an internal source. In the present case, with an incident field, it is not possible to choose as unknown the total field (increasing in one of the absorbing layers) while choosing the diffracted field would lead to complicated source terms in the perturbed part of the waveguide. We propose an approach, which consists in choosing as unknown u, the total field $u = u_{tot}$ in Ω_b, and the diffracted field $u = u_{dif}$ in $\Omega^L_\pm$. The variational formulation of the problem can then be written as:

$$\left\{\begin{array}{l} \text{Find } u \text{ such that } [u]_{\Sigma^\pm} = \mp u_{inc} \text{ and} \\ \displaystyle\int_{\Omega_b} (\sigma(u) : \varepsilon(\overline{v}) - \omega^2 u.\overline{v})dx + \\ \quad \displaystyle + \frac{1}{\alpha}\int_{\Omega^+_L \cup \Omega^-_L} (\sigma_\alpha(u) : \varepsilon_\alpha(\overline{v}) - \omega^2 u.\overline{v})dx \\ \qquad\qquad \displaystyle = \int_{\Sigma^+ \cup \Sigma^-} \sigma(u_{inc})n.\overline{v}d\gamma \\ \text{for all } v \quad \text{such that } [v]_{\Sigma^\pm} = 0 \end{array}\right. \qquad (5)$$

where n is the unit normal on $\Sigma^\pm$ exterior to Ω_b.

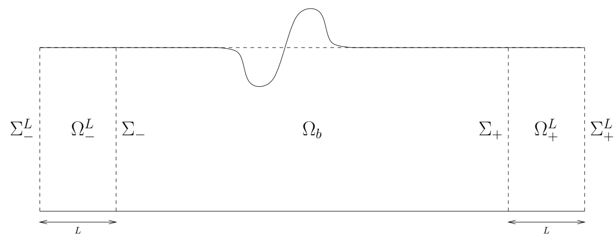

Figure 1: The truncated domain Ω^L.

Discretization

In order to approximate the solution u of (5) by finite elements, we introduce a triangular mesh of the domain Ω^L. The discretized problem consists in looking for $u \in V$ satisfying (5) for all $v \in V$, where V is a finite dimensional space composed of the displacement fields which are continuous in Ω^L and polynomial functions of degree 2 on each triangle. Using classical locally supported basis functions, this leads to solve a matricial linear system, where the matrix to be inverted is large but very sparse. Again let us point out that even if "Dirichlet-to-Neumann" like boundary conditions were available, they would lead to partially full matrices.

The computations have been done by S. Mohamed with the finite element library MÉLINA [8].

Numerical results

We consider a plate of steel whose thickness is denoted by $2h$. The velocities of longitudinal and transversal waves are given by $c_L = 5900ms^{-1}$ and $c_T = 3100ms^{-1}$. All the computations are made with a fixed mesh and for a frequency such that $\omega h/\pi = 4000ms^{-1}$. At this frequency, it can be shown that exactly 5 modes can propagate in the plate : the three symmetric modes S0, S1 and S2, and the two antisymmetric ones A0 and A1. For the sake of brevity, we will focus on the symmetric ones.

At first the method has been validated in a simple case: the physical domain is a part of the unperturbed plate ($0 < x < 35$ and $-15 < y < 15$), extended on the right by an absorbing layer ($35 < x < 40$ and $-15 < y < 15$). The displacement of the mode S0 is imposed on the lateral boundary on the left ($x = 0$ and $-15 < y < 15$), so that the exact solution is the mode S0. The H^1-norm relative error between the exact and the approximated solutions is plotted on figure 2 as a function of the modulus of the parameter α, its argument beeing fixed and equal to $-\pi/4$. This curve

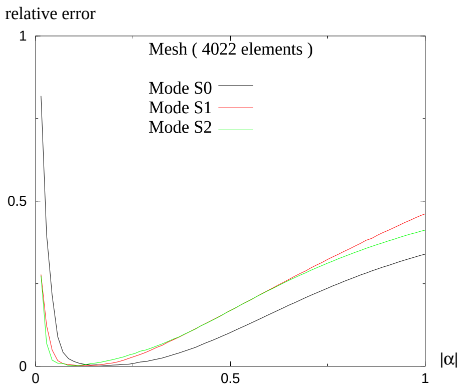

Figure 2: Error versus $|\alpha|$ for the 3 symmetric modes

presents typical features with a range of intermediate values where the error is small (only due to the finite element approximation), while the error rapidly increases for both small and large values of $|\alpha|$. Indeed, for small values of $|\alpha|$, the diffracted field becomes strongly decreasing in the layers, and the mesh is too coarse for approximating this behavior. On the other hand, when α is too large, the diffracted field is not enough decreasing in the layer, so that spurious reflexions are produced by the truncation of the layer. These effects can be clearly indentified when looking at the computed solution in the layer, as shown on figure 3.

Then the method has been applied to compute the diffraction due to a local deformation of the upper boundary of the plate when the incident wave is the propagative mode S0. The real part of the first component of the incident, total and diffracted fields are represented in the physical domain Ω_b in figure 4.

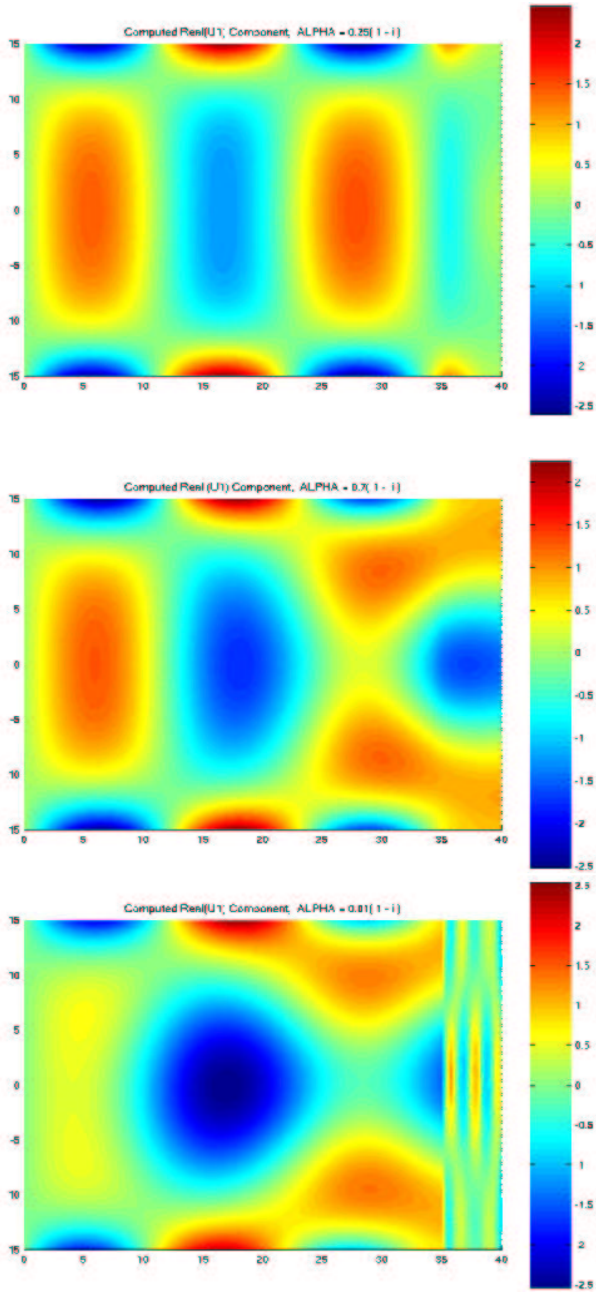

Figure 3: Simulation of the S0 mode : Real part of the first component of the solution. Top : an accurate value of $|\alpha|$, Middle: a too large $|\alpha|$, Bottom: a too small $|\alpha|$

Conclusion and perspectives

In most cases (no inverse propagative modes), PMLs are a simple and efficient tool for selecting the outgoing solution of the diffraction problem in an elastic plate. For frequencies such that inverse propagative modes exist, PMLS work neither in the frequency domain (where the outgoing solution is not well selected) nor in the time domain where they are leading to numerical instabilities. The treatment of such frequencies is an open question.

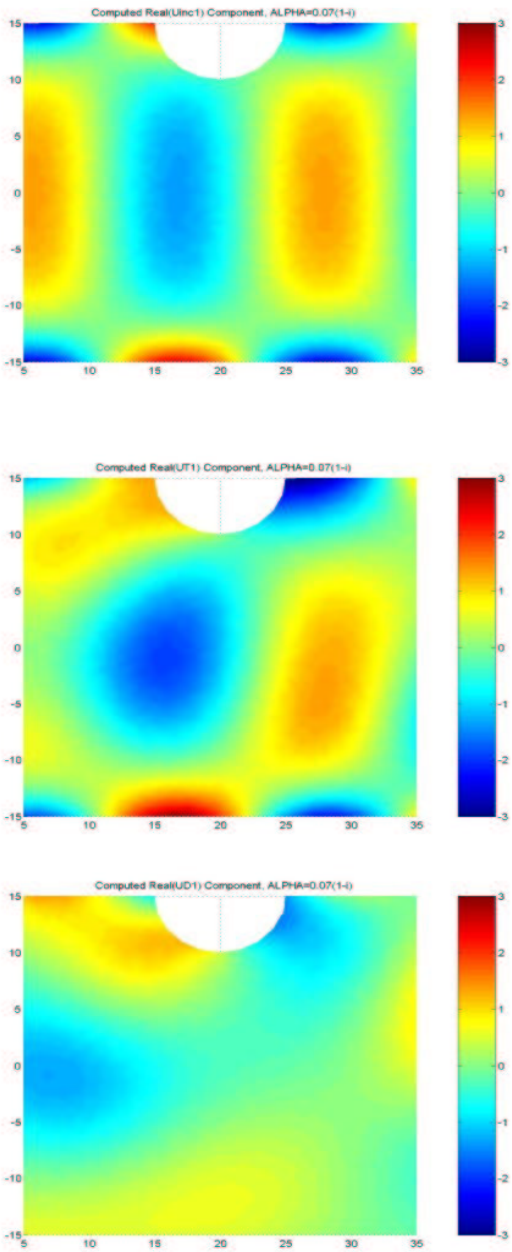

Figure 4: Top: the incident field, Middle: the total field, Bottom: the diffracted field

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