RESONANT MULTIPLE ELASTIC SCATTERING BY A FINITE NUMBER OF CYLINDRICAL CAVITIES IN AN ELASTIC MATRIX

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Abstract

Multiple scattering of elastic waves by a finite number of cylindrical cavities is studied. For very close scatterers, the regimes of interaction between the scatterers are identified and compared with those already investigated for elastic scatterers in a fluid. In case of empty cavities, a strong interferential interaction is predominant between cavities, whereas, for fluid-filled cavities, a strong resonant coupling different from that of the fluid case occurs with no obvious single scattering resonance split. The scattering *S*-matrix is also introduced. Its unitarity property, which expresses the energy conservation, allows to validate the numerical results.

Introduction

The elastic scattering by a finite number of very close scatterers is an unexplored subject from the resonance point of view. Some authors have been interested in the resonant interaction between two close elastic cylinders immersed in a fluid [1], and have shown that some low-frequency single scatterer resonances, as those of the Scholte-Stoneley, may split into new resonances whose number is in relation with the number of scatterers. In the elastic case, does such a resonant interaction exist, and, if so, is there a relation between the number of new resonances and the number of scatterers ?

In order to answer to these questions, a multiple scattering modal theory, based on that of Varadan *et al.* [2], is developed in a general way for *N* cylindrical inclusions in an elastic matrix. Numerical results are then presented for two and three identical cavities, either empty of fluid-filled. The scattering *S*-matrix is next defined in the field of modal theories by adapting the Heisenberg formalism of quantum physics [3]. Its unitarity property expresses indeed the energy conservation law and consequently allows to check numerical results.

Elastic scattering by N cylindrical cavities

A distribution of *N* cylindrical cavities in an elastic matrix is considered. The longitudinal *L* and transverse *T* waves propagate with phase velocities c_L and c_T. The scatterers are parallel and infinite in the *Oz* direction (cf. Fig. 1), and a plane harmonic wave propagating in the *(Oxy)* plane is considered. It is thus a two dimensional problem. In the incidence plane, O_i is the center of the i*th* scatterer, (d_i,χ_i) and (r_{ij},θ_{ij}) are the polar coordinates of O_i with respectively O and O_j as origin. (r,θ) and (r_i,θ_i) are the coordinates of the observation point *P* in the coordinates systems centered respectively on *O* and O_i. One considers an incident *L* or *T*-wave of unit-amplitude, angular frequency ω, incidence angle α_{inc} and wave number $k_{inc}=\omega/c_{inc}$ (the index *inc*, which stands for *L* and *T*, refers to the incident wave). In the coordinates system (r_i,θ_i), the incident potential displacement may be written as

$$\begin{aligned}\varphi_{inc}^{(i)} &= e^{ik_{inc}d_i\cos(\chi_i-\alpha_{inc})}\sum_{n=-\infty}^{+\infty} i^n e^{-in\alpha_{inc}} J_n\left(k_{inc}r_i\right)e^{in\theta_i} \\ &= \sum_{n=-\infty}^{+\infty} A_n^{(i)inc} J_n\left(k_{inc}r_i\right)e^{in\theta_i},\end{aligned} \tag{1}$$

where J_n is the Bessel function of order *n*. The time factor $exp(-i\omega t)$ is omitted in (1) as well as everywhere throughout the paper.

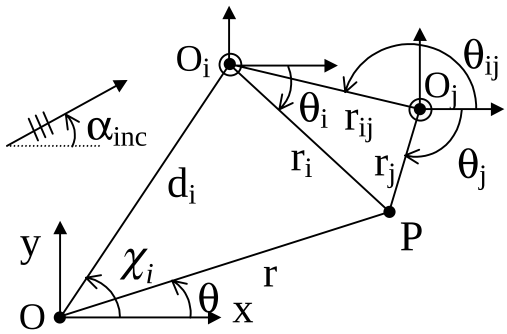

Figure 1 : Elastic scattering by *N* cylindrical cavities : geometry

The acoustic field scatterered by the i*th* scatterer can be expressed as

$$\phi_S^{(i)} = \sum_{n=-\infty}^{+\infty} C_n^{(i)incL} H_n^{(1)}\left(k_L r_i\right)e^{in\theta_i}, \tag{2}$$

$$\psi_S^{(i)} = \sum_{n=-\infty}^{+\infty} C_n^{(i)incT} H_n^{(1)}\left(k_T r_i\right)e^{in\theta_i}, \tag{3}$$

with $H_n^{(1)}$ the Hankel function of the first kind. In the present part, the problem consists in determining the unknown scattering coefficients $C_n^{(i)incL}$ and $C_n^{(i)incT}$. The incident field on the i*th* scatterer is the sum of the plane incident wave and of the field scattered from all others scatterers. Thus, assuming the scattering coefficients $T_m^{(i)LL}$, $T_m^{(i)LT}$, $T_m^{(i)TL}$ and $T_m^{(i)TT}$ of the single cavity are known, the field scattered by the i*th* cavity can be rewritten by means of all other scattering coefficients $C_n^{(j\neq i)incL}$ and $C_n^{(j\neq i)incT}$ [2]. For instance, for an incident *L*-wave, (2) becomes

$$\phi_S^{(i)} = \sum_{m=-\infty}^{+\infty} T_m^{(i)LL} (A_m^{(i)L} + \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{n=-\infty}^{+\infty} G_{Lnm}^{(ji)} C_n^{(j)LL}) H_m^{(1)}(k_L r_i) e^{im\theta_i}$$
$$+ \sum_{m=-\infty}^{+\infty} T_m^{(i)TL} \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{n=-\infty}^{+\infty} G_{Tnm}^{(ji)} C_n^{(j)LT} H_m^{(1)}(k_L r_i) e^{im\theta_i}, \quad (4)$$

with operator $G_{L,T\,nm}^{(ji)}$ defined as

$$G_{L,T\,nm}^{(ji)} = e^{i(n-m)\theta_{ij}} H_{n-m}^{(1)}\left(k_{L,T} r_{ij}\right). \quad (5)$$

Identification of the two different expressions of the scattered field leads finally to the following general linear system

$$\begin{cases} C_m^{(i)incL} - T_m^{(i)LL} \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{n=-\infty}^{+\infty} G_{Lnm}^{(ji)} C_n^{(j)incL} \\ \quad - T_m^{(i)TL} \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{n=-\infty}^{+\infty} G_{Tnm}^{(ji)} C_n^{(j)incT} = T_m^{(i)incL} A_m^{(i)inc}, \\ C_m^{(i)incT} - T_m^{(i)LT} \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{n=-\infty}^{+\infty} G_{Lnm}^{(ji)} C_n^{(j)incL} \\ \quad - T_m^{(i)TT} \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{n=-\infty}^{+\infty} G_{Tnm}^{(ji)} C_n^{(j)incT} = T_m^{(i)incT} A_m^{(i)inc}. \end{cases} \quad (6)$$

After solved it, the total scattered field is obtained by summing all the scattered potentials $\phi_S^{(i)}$ and $\psi_S^{(i)}$, and can be approximated in far field ($k_{L,T} r \gg 1$) by use of the Hankel function asymptotic development. For instance, the *L*-component of the total scattered field may be written in far field as

$$\phi_S = f^{incL}(\alpha_{inc}, \theta) \frac{e^{i\left(k_L r - \frac{\pi}{4}\right)}}{\sqrt{k_L r}}, \quad (7)$$

where the following far field scattering amplitude is introduced

$$f^{incL}(\alpha_{inc}, \theta) = \sum_{i=1}^{N} \sum_{n=-\infty}^{+\infty} C_n^{(i)incL} e^{-ik_L d_i \cos(\chi_i - \theta)} e^{in\left(\theta - \frac{\pi}{2}\right)}. \quad (8)$$

The computation of the scattering amplitudes requires many calculations. Thus, to validate the results, the energy conservation law has been verified thanks to the unitarity property of the scattering S-matrix whose formulation is given below.

Scattering matrix formalism

To introduce the scattering *S*-matrix, the first step consists in expressing the total acoustic field in far field. Considering an incident *L*-wave and using the Bessel function asymptotic expansion, one obtains the following expression

$$\Phi = \frac{1}{\sqrt{2\pi}} \left\{ \sum_{n=-\infty}^{+\infty} (-1)^n e^{in(\theta - \alpha_L)} \frac{e^{-i\left(k_L r - \frac{\pi}{4}\right)}}{\sqrt{k_L r}} + \sum_{n=-\infty}^{+\infty} e^{in(\theta - \alpha_L)} \frac{e^{i\left(k_L r - \frac{\pi}{4}\right)}}{\sqrt{k_L r}} \right\} + f^{LL}(\alpha_L, \theta) \frac{e^{i\left(k_L r - \frac{\pi}{4}\right)}}{\sqrt{k_L r}}, \quad (9)$$

$$\Psi = f^{LT}(\alpha_L, \theta) \frac{e^{i\left(k_T r - \frac{\pi}{4}\right)}}{\sqrt{k_T r}}. \quad (10)$$

Here, Ψ is a sum of outgoing *T*-waves, whereas, on account of the asymptotic expansion of the incident wave, Φ is a sum of outgoing and incoming *L*-waves. According to Landau and Lifchitz in quantum physics [3], some more general scattering process can be generated from linear combinations of (9) and (10) with a continuous variation of α_L. Thus, let us introduce the scalar product

$$\left\langle f^{LL}, F \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} f^{LL}(\alpha_L, \theta) F^*(\alpha_L) \, d\alpha_L, \quad (11)$$

where $F(\alpha_L)$ is a 2π-periodic function defined by the Fourier's series development

$$F(\alpha_L) = \sum_{n=-\infty}^{+\infty} F_n e^{in\alpha_L}. \quad (12)$$

Therefore, the scattered fields become

$$\Phi = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\left(k_L r - \frac{\pi}{4}\right)}}{\sqrt{k_L r}} F^*(\pi - \theta) + \frac{1}{\sqrt{2\pi}} \frac{e^{i\left(k_L r - \frac{\pi}{4}\right)}}{\sqrt{k_L r}} \hat{S}^{LL}\left[F^*(\alpha_L)\right](\theta), \quad (13)$$

$$\Psi = \frac{1}{\sqrt{2\pi}} \frac{e^{i\left(k_T r - \frac{\pi}{4}\right)}}{\sqrt{k_T r}} \hat{S}^{LT}\left[F^*(\alpha_L)\right](\theta), \quad (14)$$

where operators $\hat{S}^{LL}$ and $\hat{S}^{LT}$ describe the scattered *L* and *T*-waves generated by the incident *L*-wave. They act on $F^*(\alpha_L)$ as follows

$$\hat{S}^{LL}\left[F^*(\alpha_L)\right](\theta) = F^*(\theta) + \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f^{LL}(\alpha_L, \theta) F^*(\alpha_L) \, d\alpha_L, \quad (15)$$

$$\hat{S}^{LT}\left[F^*(\alpha_L)\right](\theta) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f^{LT}(\alpha_L, \theta) F^*(\alpha_L) \, d\alpha_L. \quad (16)$$

Of course, proceeding as above with an incident T-wave leads to two other operators $\hat{S}^{TL}$ and $\hat{S}^{TT}$. As part of modal theories, one has to project the scattering amplitudes on the basis $\{e^{-ip\alpha}\}$ (p þ). Calculating the projection on this basis is equivalent to determine the scattering matrix elements S_{pq} from the following relation

$$\hat{S}\left[e^{ip\alpha}\right](\theta) = \sum_{q=-\infty}^{+\infty} S_{pq} e^{iq\alpha} . \tag{17}$$

Since there are four scattering operators, the projection leads to four infinite matrices whose elements are given by

$$\begin{aligned} S_{pq}^{l_1 l_2} &= \delta_{l_1 l_2}\delta_{pq} \\ &+2\sum_{j=1}^{N}\sum_{n=-\infty}^{+\infty} i^{-q}\left\langle C_n^{(j)l_1 l_2}, e^{-ip\alpha_{l_1}}\right\rangle e^{-i(q-n)\chi_j} J_{q-n}\left(k_{l_2} d_j\right), \end{aligned} \tag{18}$$

where l_1 and l_2 stand for L or T.

From these matrices, the scattering S-matrix can be built so that the energy law conservation SS^{+}= í (the symbol + designates the Hermitian product while í is the identity matrix) is verified. In fact, energy flow conservation calculations [4] clearly show that there are several possible constructions. For instance

$$S = \begin{bmatrix} S^{LL} & S^{LT} \\ S^{TL} & S^{TT} \end{bmatrix} \tag{19}$$

with

$$S^{l_1 l_2} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & S^{l_1 l_2}_{p\text{-}1p\text{-}1} & S^{l_1 l_2}_{p\text{-}1p} & S^{l_1 l_2}_{p\text{-}1p+1} & \cdots \\ \cdots & S^{l_1 l_2}_{pp\text{-}1} & S^{l_1 l_2}_{pp} & S^{l_1 l_2}_{pp+1} & \cdots \\ \cdots & S^{l_1 l_2}_{p+1p\text{-}1} & S^{l_1 l_2}_{p+1p} & S^{l_1 l_2}_{p+1p+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{20}$$

The unitarity of S has been numerically checked for two and three cavities with different spatial configurations.

Numerical results and discussion

The scattering by several identical cavities is studied when the distance between scatterers varies. The computations of scattering amplitudes have been performed for an aluminum matrix with the parameter values : ρ = 2700 kg/m^3, c_L = 6380 m/s and c_T = 3140 m/s. It should be noted that all curves are plotted versus the reduced frequency $x_L = k_L a$ (a is the radius of the cavities), and that all results have been obtained in the backscattering case ($\theta = \alpha_L - \pi$). As this study is based on the detection of resonances, we have chosen to present only the derivative of the scattering amplitude phases. The resonance frequencies are then merely identified by pointing out the maxima of the phase derivative. As the resonant behavior of empty and fluid-filled cavities differs, the two types of cavity are considered separately.

Scattering by empty cavities

The resonant behavior of an empty cavity excited by L and T-waves is already known. In both cases, the waves circumnavigating the cavity are too much attenuated to resonate and to affect fundamentally the scattered waves. Depending on their own polarization, they interfere either with the L-waves or the T-waves that are specularly reflected by the cavity. As the Rayleigh wave, which is excited by T-waves, is the least attenuated, we study here only the $T \rightarrow T$ scattering.

Let us consider now two empty cavities in an eclipse configuration, i.e. when the line linking the centers of the cavities is parallel to the propagation direction of incident wave. The centers are separated by distance βa. Fig. 2 presents the evolution of the peaks of the phase derivative (in solid lines) as x_L and β vary. The resonance frequencies of the single cavity are presented in horizontal dotted lines.

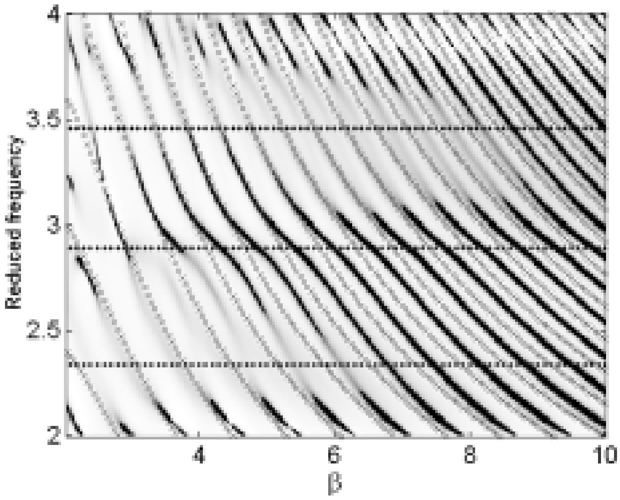

Figure 2 : Phase derivative modulus for $T \rightarrow T$ scattering by two empty cavities in the eclipse configuration, plotted versus β and x_L

We observe the emergence of a great number of peaks which belong to a family of parabolic curves. These curves emphasize interferences between the specular echo (see Fig. 3, trajectory A) and the waves that propagate back and forth from one cavity to the other one (trajectory B). Such an interferential interaction has already been observed and analyzed for two elastic shells immersed in water [5]. To confirm this interpretation, we have assumed that constructive interferences occur when the distance that separates trajectory B from trajectory A corresponds to an integer number of wavelengths. The resulting curves are superimposed (parabolic dotted lines) on the diagram, and we notice a good agreement with the evolution of the phase derivative maxima.

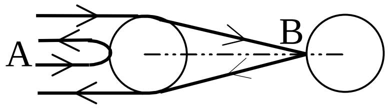

Figure 3 : Propagation paths of the waves that interfer

This result confirms that the interaction regime between the two cavities is not resonant, but always interferential, even for very small values of β.

Scattering by water-filled cavities

The resonant behavior of water-filled cavities is governed by the Whispering-Gallery waves which are much less damped than the Raleigh and Franz waves. Therefore, the single cavity exhibits a resonant behavior, contrary to the empty one, and a resonant coupling may be observed when N ($N>1$) cavities are close to each other To illustrate this strong resonant character, let us consider three close cavities ($\beta =2.1$) in an equilateral triangle configuration, for example. The diagram in Fig. 3 presents the evolution of the phase derivative peaks versus x_L and the incidence angle α_L for $L\rightarrow L$ scattering. The case $\alpha_L =0°$ corresponds to an incidence on the triangle apex. The three resonance frequencies corresponding to the single water-filled cavity are superimposed (in dotted lines) on the diagram.

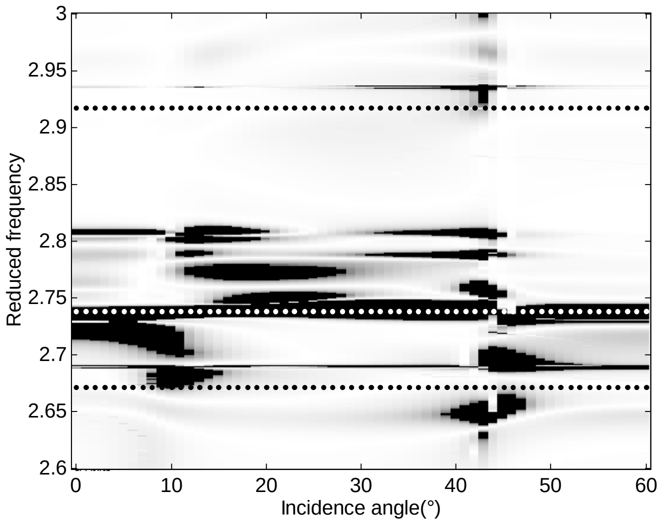

Figure 4 : Phase derivative modulus for $L\rightarrow L$ scattering by three water-filled cavities in an equilateral configuration plotted versus θ and α_L, diagram obtained for $\beta =2.1$

We notice that a large number of resonances appears in the phase derivative plot (for instance 10 at $\alpha_L =43°$). The resonances frequencies do not change versus α_L, which is characteristic of a resonant behavior. In comparison to the case of shells immersed in water, for example, we note that there is no obvious relation between the number of scatterers and the number of resonances. Moreover, most of the resonances are far off those related to the single cavity scattering and do not seem to result from the split of any particular one. These results show that the resonant interaction between scatterers is more complicated in an elastic medium than in a fluid medium. Even for large values of β ($\beta=10$), this strong resonant coupling between cavities still exists. The regime of interferential interaction observed for empty cavities appears only for β values greater than 10.

Conclusion

The scattering of an elastic wave by cylindrical cavities in an elastic matrix has been studied. In particular, we wished to know if a resonant coupling similar to the one existing for scatterers immersed in a fluid could occur in the elastic case. To this end, the general multiple scattering formalism for N inclusions has been introduced, as well as the scattering S-matrix, whose unitarity property expresses the energy conservation and allows us to validate computations. Two different regimes of interaction between the cavities have been enlightened : interferential and resonant. For empty cavities, the interferential interaction is predominant, even for very close scatterers. In the case of water-filled cavities, a strong resonant coupling occurs between the cavities, even when they are very distant from each other. Because of the appearance of a considerable number of resonances relatively to the number of scatterers, among other phenomenon, understanding of the resonant coupling appears to be much more complicated than in the fluid case.

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