# THE PHASE GRADIENT METHOD (PGM) APPLIED TO VISCOELASTIC PLATES

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**Abstract**

It was shown for elastic plates immersed in water that the PGM allows us to obtain a radiation quality factor $Q_R$ [1] by studying the frequency derivative of the phase $\phi$ of the reflection coefficient. Indeed, when plotted versus frequency in the vicinity of a resonance, the product of the frequency with the frequency phase derivative, $f\,\partial\phi/\partial f$, exhibits a maximum located at the resonance frequency. The maximum amplitude is proportional to the ratio of the resonance frequency to the resonance half-width, which corresponds to the definition of a quality factor. In this paper, we study a viscoelastic plate in which the longitudinal and transverse phase velocities are complex. We show that the PGM is still efficient to obtain a quality factor $Q_R$. Moreover, this method allows us to decompose it as follows: $1/Q_R = 1/Q_E + 1/Q_V$, where $Q_E$ depends on the elastic properties and $Q_V$ on the viscous ones.

**Introduction**

A radiation $Q_R$ factor can be evaluated by studying the frequency derivative of the phase $\phi$ of the reflection coefficient $\underline{R}$ of a plate in the vicinity of a resonance frequency. This $Q_R$ factor is linked to a frequency pole of $\underline{R}$, denoted as $\underline{f}_P = f_{res} - j\ \Gamma/2$, and defined as $Q_R = f_{Res}/\Gamma$. It is associated with the temporal attenuation of the reflected wave by the plate. For an elastic plate, the resonance width depends on the plate elastic parameters and on the fluid loading. In the case of a viscoelastic plate, the resonance width also depends on the viscosity in the plate. This global width $\Gamma$ can be decomposed in a term $\Gamma_E$ due to the elastic effects and a term $\Gamma_V$ due to the viscous ones [2]. It implies that $\Gamma = \Gamma_E + \Gamma_V$ and therefore the global radiation $Q_R$ factor can be obtained from the following relation: $1/Q_R = 1/Q_E + 1/Q_V$, where $Q_E = f_{Res}/\Gamma_E$ is an elastic Q factor and $Q_V = f_{Res}/\Gamma_V$ a viscosity Q factor. The $Q_R$ factor can be obtained from the roots of the dispersion equation of the immersed plate, the $Q_V$ factor from the roots of the dispersion equation of the plate in vacuum and the $Q_E$ factor from the roots of the dispersion equation of the immersed plate without viscosity. In this paper, we show that the study of parts of the global function $f\,\partial\phi/\partial f$ derived from the PGM, using adapted corrective terms may give good estimates of the different Q factors introduced.

**Basis of the method**

The factorized expression of the reflection coefficient is [3]:

$$\underline{R} = \frac{\underline{N}}{\underline{A}\,\underline{S}} = \frac{\underline{C}_A \underline{C}_S - \underline{\tau}^2}{(\underline{C}_A + j\underline{\tau})(\underline{C}_S - j\underline{\tau})}.$$

The roots of $\underline{C}_{A,S}$ (resp. $\underline{A}$, $\underline{S}$) correspond to the antisymmetric and symmetric normal modes of the plate in vacuum (resp. in water) and $\underline{\tau}$ is the ratio of the acoustic impedances in water and in the plate. The functions $\underline{C}_{A,S}$ and $\underline{\tau}$ are complex for a viscoelastic plate because the longitudinal and transverse phase velocities are complex. They can be written

$$\underline{c}_{L,T} = \frac{c_{L,T_0}}{1 + j\, r_{L,T}}$$

where $c_{L,T_0}$ are the phase velocities without absorptions and $r_{L,T}$ are the loss factors defined as $r_{L,T} = \alpha_{L,T} c_{L,T_0}/\omega$. $\alpha_{L,T}$ are the absorption coefficients corresponding to the imaginary parts of the wave numbers. They are assumed linearly dependent on the angular frequency $\omega$ in the frequency range 2 MHz < f< 3 MHz. So, the loss factors are constant. For the numerical results, we consider a 3 mm-thick makrolon plate immersed in water experimentally investigated [4]. The parameter values are $c_{L_0} = 2235$ m/s (measured), $c_{T_0} = 800$ m/s (estimated), $r_L = 0.0196$ (measured), $r_T = 0.1$ (estimated), density $\rho_S = 1200$ kg/m$^3$ for the plate and the phase velocity is $c_F = 1470$ m/s and density is $\rho_F = 1000$ kg/m$^3$ for water.

The global phase $\phi$ of the reflection coefficient can be written as: $\phi(f) = \phi_{\underline{N}}(f) - \phi_{\underline{A}}(f) - \phi_{\underline{S}}(f)$, where

$$\phi_{\underline{X}}(f) = \operatorname{atan}\left(\frac{X_I(f)}{X_R(f)}\right),\ \underline{X} = \underline{N}, \underline{A}, \underline{S}.$$

Here and in the following, the indices R and I indicate the real or imaginary part of $\underline{X}$. the functions depending on the frequency derivatives of the different phases can be written

$$f\frac{\partial\phi_{\underline{X}}}{\partial f} = f\frac{X_R\frac{\partial X_I}{\partial f} - X_I\frac{\partial X_R}{\partial f}}{X_R^2+X_I^2}.$$

In Figure 1, the plots of these functions are presented at $\theta = 5°$. Contrary to the case of an elastic plate, the global phase derivative (green line) exhibits minimums in the vicinity of the resonances and we can observe a non zero background. The phase derivative of the numerator $\underline{N}$ (blue line) is negligible. In the vicinity of a resonance associated with a symmetric (resp. antisymmetric) mode, the phase derivative of $\underline{S}$ (red line) (resp. $\underline{A}$) exhibits a maximum and the phase derivative of $\underline{A}$ (black line) (resp. $\underline{S}$) a minimum.

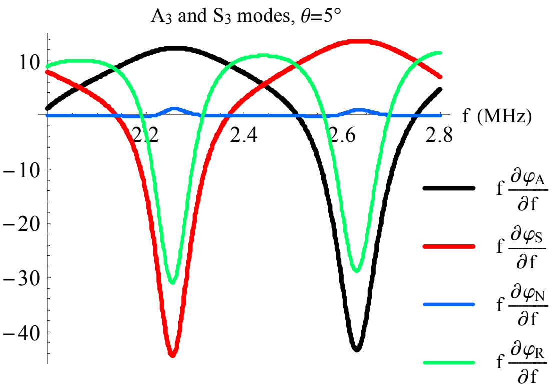


Figure 1.

*Determination of the $Q_R$ factor*

We can compare the plots of $f\,\partial\phi_{\underline{A},\underline{S}}/\partial f$ with the ones of the approximate function of Breit-Wigner type defined as $(f\,\partial\phi/\partial f)_{app} = \frac{f_{Res}\,\Gamma/2}{(f-f_{Res})^2+(\Gamma/2)^2}$, where $f_{Res} - j\Gamma/2$ is a root of $\underline{A}$ or $\underline{S}$. In Figure 2, we present the plots of the exact functions $f\,\partial\phi_{\underline{A}}/\partial f$ (black line) in the vicinity of the $A_3$ mode and $f\,\partial\phi_{\underline{S}}/\partial f$ (red line) in the vicinity of the $S_3$ mode, as well the plots of the approximate functions $(f\,\partial\phi_{\underline{A}}/\partial f)_{app}$ (dashed blue line) and $(f\,\partial\phi_{\underline{S}}/\partial f)_{app}$ (dashed green line). Even if the exact plots of $f\,\partial\phi_{\underline{A},\underline{S}}/\partial f$ do not fit a resonant Breit-Wigner form in the vicinity of a resonance due to an antisymmetric or symmetric mode, we observe in Figure 2 that their maximums nearly coincide with those of the approximate functions. These maximums are located at the resonance frequencies and their amplitudes are about twice the value of the $Q_R$ factors. The error for the $A_3$ and $S_3$ modes is inferior to 7 %.

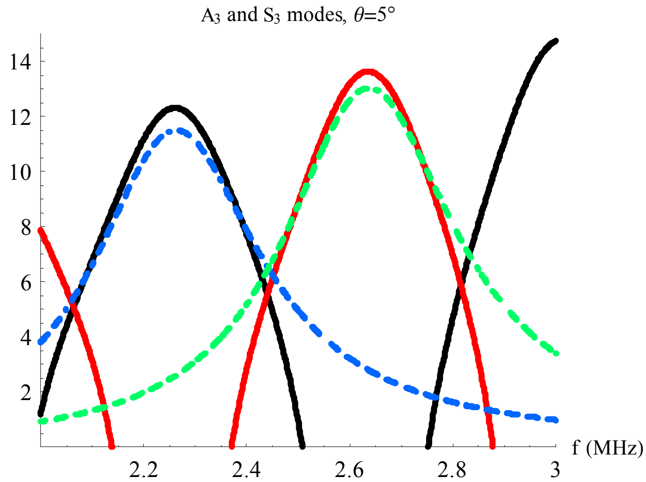


Figure 2.

So, the PGM give a good estimate of the global radiation $Q_R$ factor.

*Determination of the $Q_E$ factor*

In order to obtain the elastic $Q_E$ factor, we have plotted $\partial\phi_{\underline{A},\underline{S}}/\partial f$ considering no longitudinal loss factor in the plate in Figure 3. Due to the strong transverse loss factor ($r_T$ = 0.1), all happens as if the plate was fluid.

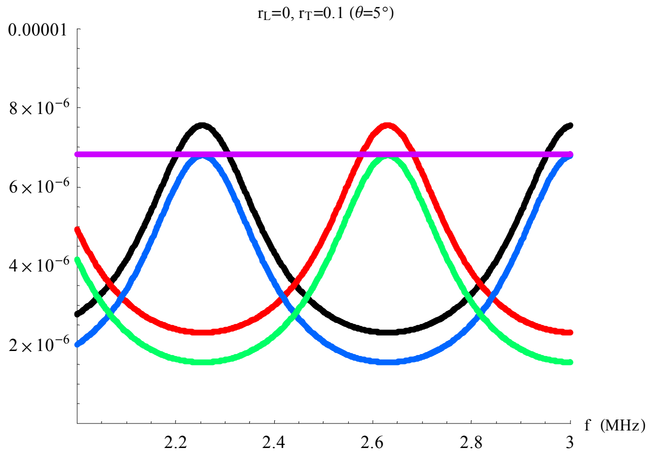


Figure 3.

We observe maximums regularly spaced of same amplitude, alternately associated with antisymmetric (black line) and symmetric (red line) mode resonances. The horizontal line indicates the inverse of the imaginary parts of the roots $\underline{f}_E = f_E - j\Gamma_E/2$ of $\underline{A}$ and $\underline{S}$ with $r_L$ = 0. We note that the maximums underestimate the value of the imaginary part. For the fluid plate model, we have an analytical expression of the roots of the dispersion equation:

$$\underline{f}_E = c_{L_0}\Big/ d\sqrt{1-(c_{L_0}/c_F\sin\theta)^2}\,(n\pi - j\,a\tanh\tau),$$

where n is an integer. Analytically, the inverse of the maximum amplitude of $\partial\phi_{\underline{A},\underline{S}}/\partial f$ is:

$$c_{L_0}\Big/\pi d\sqrt{1-\left(c_{L_0}/c_F\sin\theta\right)^2}\ \tau$$

In our case, the acoustic impedance ratio $\tau$ is not small. For an incidence angle $\theta$ ranging from 0° to 20°, its value is about 0.5. So, $a\tanh\,\tau \cong \tau + 1/3\,\tau^3 + \ldots$ cannot be identified to $\tau$ and we have to consider the third order term. In order to obtain the true value of the root imaginary part via the PGM, we have to substract to the function $\partial\phi_{\underline{A},\underline{S}}/\partial f$ the correction term $\pi d\sqrt{1-\left(c_{L_0}/c_F\sin\theta\right)^2}\Big/c_{L_0}\ \tau/3$.

As shown in Figure 3, the maximum amplitudes of the corrected functions (blue line for antisymmetric modes and green line for symmetric ones)) allow us to obtain the root imaginary parts. Multiplied by the frequency, the maximum amplitudes allow to estimate the elastic $Q_E$ factor.

*Determination of the $Q_V$ factor*

The viscosity $Q_V$ factor can be obtained from the roots $\underline{f}_V = f_V - j\Gamma_V/2$ of the dispersion equations $\underline{C}_{A,S} = 0$ of the viscoelastic plate in vacuum. If we still consider the viscoelastic solid plate as a fluid plate, the function $\underline{C}_S$ is the inverse of the function $\underline{C}_A$. It can be written

$\underline{C}_S = \underline{C}_A^*\big/\left|\underline{C}_A\right|^2$ (the asterisk * indicates the complex conjugate). The exact phase of the $\underline{S}$ function is

$$\phi_{\underline{S}} = a\tan\frac{C_{S_I}-\tau_R}{C_{S_R}+\tau_I}.$$

If we neglect the second order terms, the approximate expression can be written

$$\phi_{\underline{S}_{app}} = -a\tan\frac{C_{A_I}}{C_{A_R}}.$$

Using the RST expansion in the vicinity of the real part $f_V$ of a root of $\underline{C}_A$ and identifying the half-width $\Gamma_V/2$ to $-\left(C_{A_I}/\partial C_{A_R}/\partial f\right)_{f_V}$, we can write the function $f\,\partial\phi_{\underline{S}_{app}}/\partial f$ in the following resonant form of Breit-Wigner type:

$$f\,\partial\phi_{\underline{S}_{app}}/\partial f = -f_V\,\Gamma_V/2\Big/\left(\left(f-f_V\right)^2+\left(\Gamma_V/2\right)^2\right).$$

It explains why the plot of the exact function $f\,\partial\phi_{\underline{S}}/\partial f$ in Figure 1 exhibits a minimum for the resonance frequency assigned to the $A_3$ mode. Due to the small value of the acoustic impedance ratio, the exact function contains a background term to remove in order to only keep the resonant term. The background can be identified to the function $f\,\partial\phi_{\underline{S}(r_L=0)}/\partial f$ in which no longitudinal loss factor $r_L$ is taken into account.

In Figure 4, we have plotted, in the vicinity of the $A_3$ mode resonance frequency ($\theta = 5°$), the exact function $f\,\partial\phi_{\underline{S}}/\partial f$ (black line), the function $f\,\partial\phi_{\underline{S}(r_L=0)}/\partial f$ (red line), the difference of the two previous functions, namely the corrected function (blue line) and the function $BWC_A$. This last function is defined as

$$BWC_A = -f_V\,\Gamma_V/2\Big/\left(\left(f-f_V\right)^2+\left(\Gamma_V/2\right)^2\right),$$

where $f_V$ and $\Gamma_V/2$ are the real and imaginary parts of the zero of $\underline{C}_A$ corresponding to the $A_3$ mode.

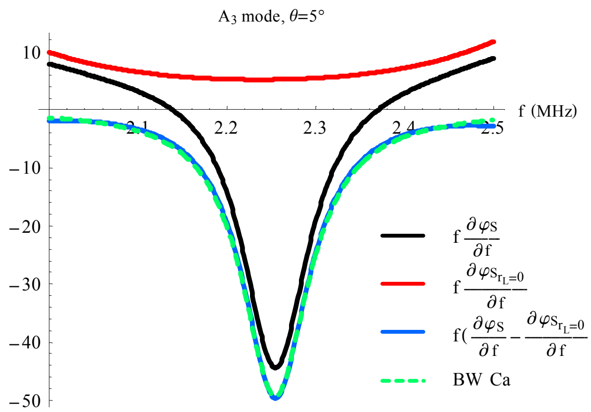


Figure 4

We observe that the plots of the corrected function and the $BWC_a$ function nearly coincide.

Therefore, the study of the corrected function $f\,\partial\phi_{\underline{S}}/\partial f$ allows us to obtain the $Q_V$ factor for an antisymmetric mode. As well, the study of the corrected function $f\,\partial\phi_{\underline{A}}/\partial f$ allows us to obtain the $Q_V$ factor for a symmetric mode.

In the following, we verify the validity of the obtaining of the global quality factor $Q_R$ via the determination of the elastic $Q_E$ factor and the viscosity $Q_V$ factor either by the roots of the adapted dispersion equations or by the PGM. This validation is performed for a given mode ($A_3$ mode) when the incidence angle varies from 0° to 20°. In this angle range, the resonance frequencies of the $A_3$ mode ranges from 2.2 MHz to 2.7 MHz. In Figure 5, we compare the frequency evolutions of the $Q_R$, $Q_E$ and $Q_V$ factors obtained either from the calculations of the roots of the associated dispersion equations or from the study of the frequency phase derivatives, corrected or not.

The plots of the evolutions of the $Q_V$ factor obtained from the roots of the dispersion equation $\underline{C}_A$ (solid orange line) and the $Q_V$ factor obtained from the corrected $f\,\partial\phi_{\underline{S}}/\partial f$ function (dashed yellow line) nearly coincide. They linearly decrease from about 51 to 43. The plots of the evolutions of the $Q_E$ factor obtained from the roots of the dispersion equation $\underline{A}(r_L = 0)$ (solid blue line) and the $Q_V$ factor obtained from the corrected $f\,\partial\phi_{\underline{A}(r_L=0)}/\partial f$ function (dashed green line) are superimposed. They slightly decrease from 15.5 to 14.5. The evolution of the $Q_R$ factor obtained from the roots of the dispersion root $\underline{A}$ is shown by the solid black line, the one of the $Q_R$ factor obtained from the $f\,\partial\phi_{\underline{A}}/\partial f$ function is shown by the dashed red line. These last two plots do not coincide as well as the two first couples of plots, but the difference is always inferior to 7 %. They decrease from 12 to 10.8 very smoothly.

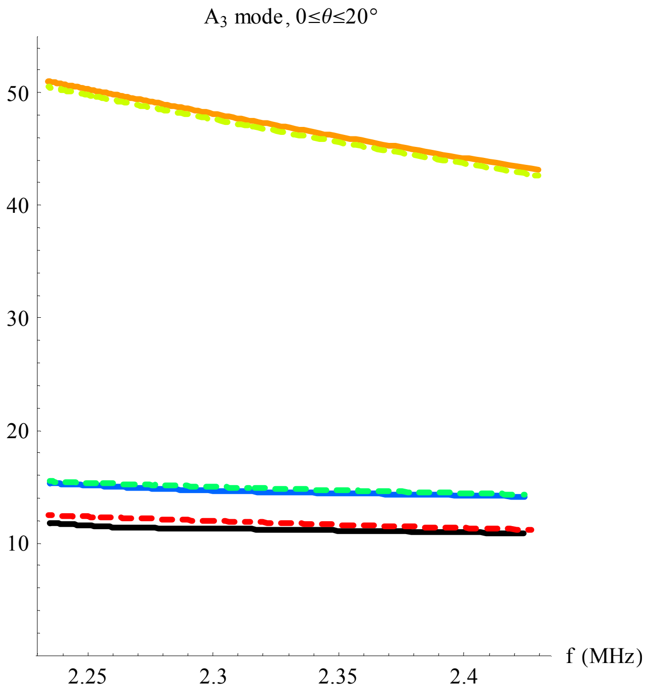


Figure 5

We can say that the comparison of the results obtained by the two methods are globally correct. We can also observe that the main part of the global radiation quality factor $Q_R$ is due to the elastic $Q_E$ factor. Nevertheless, the influence of the viscosity featured by the $Q_V$ factor is not negligible.

The $Q_R$ factor can be recovered from the $Q_E$ and $Q_v$ factor by means of the relation $Q_R = Q_E Q_V/(Q_E + Q_V)$. In Figure 6, we compare the plots of the evolution of the $Q_R$ factor obtained from the roots of the dispersion equation $\underline{A}$ (solid black line), the evolution of the $Q_R$ recovered from the $Q_E$ and $Q_V$ factors obtained from the roots of the associated dispersion equations (dashed red line) and the evolution of the $Q_R$ recovered from the $Q_E$ and $Q_V$ factors obtained from the phase derivatives (dashed blue line). The comparison of the three plots is quite good.

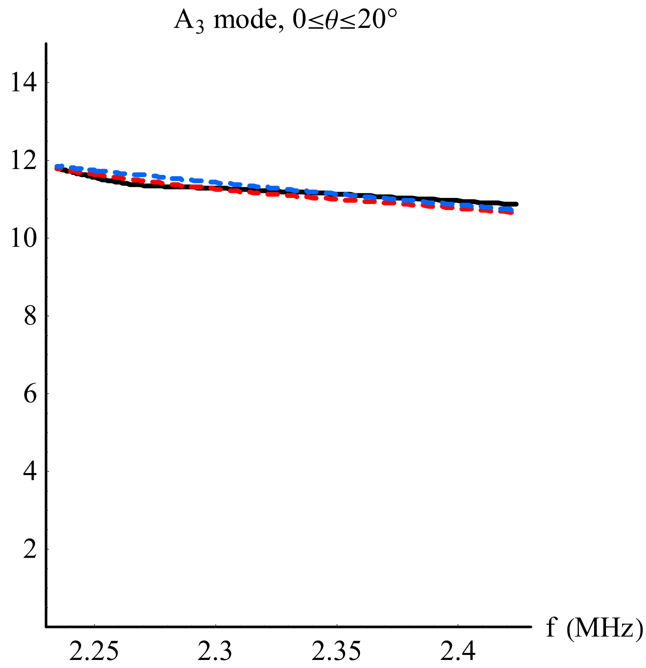


Figure 6

**Conclusion**

The PGM is a well adapted method to obtain the radiation quality factor $Q_R$ associated with the resonances of a viscoelastic plate. It gives good estimates of this Q factor whose exact determination needs heavy calculations of complex roots of the dispersion equation. Moreover, the study of the properties of the frequency derivatives of the different parts of the phase of the reflection coefficient permits to separate the elastic and viscous parts involved in the radiation mechanism by the viscoelastic plate.

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