ACTIVE CONTROL OF HARMONICS IN NONLINEAR ACOUSTIC RESONATORS

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Abstract

Consistent with the second-order nonlinear theory, acoustic fields in the resonator can be represented by counter-propagating waves which are assumed to not interact in the resonator volume. These waves are coupled only by boundary conditions. If we suppose that the waves are slowly varying in space and in time it is possible to describe the waves by means of the inhomogeneous Burgers equation. When the exciting piston radiates more than one eigen-frequency of the resonator one can control generation of harmonics. Assuming that the piston radiates only the fundamental and second eigen-frequency it is possible to get the Whittaker-Hill equation by means of the Cole-Hopf transformation for steady-state waves. New asymptotic solutions of this equation are presented in this work. The approximate solutions, which enable to describe nonlinear standing waves in the resonator, are compared with numerical ones and their validity limits are also discussed in this paper.

Introduction

Nowadays, we can observe the renewed interest in nonlinear standing waves. Frequently, the application of nonlinear standing waves is connected with the high quality resonators. These resonators enable to accumulate large amount of acoustic energy. A number of devices uses the high quality resonators because the sound waves are so powerful in these resonators that they can potentially carry out tasks which conventionally require mechanical moving parts in current technologies (e.g. the acoustic compressors). In addition the high quality resonators enable to increase efficiency of the thermoacoustic engines (e.g. acoustic refrigerators), see e.g. [1].

However, using of nonlinear standing waves is limited by the nonlinear attenuation that causes the acoustic saturation effects. The nonlinear attenuation suppression enables to increase the quality factor of the resonators.

This paper deals with the method of the active second harmonic suppression mentioned above both analytically and numerically. It is known that for this case it is possible to describe generation of the higher harmonics by means of the inhomogeneous Burgers equation. The resonator is driven by a piston whose motion is characterized by two superposed sinusoidal motions. This problem was treated for stationary state in paper [3]. However, authors of these papers took into account only inviscid solutions. It means that discontinuities were contained in their solutions. Unlike these solutions we present new asymptotic solutions which take into account dissipative effects. Some of the solutions are also presented in the spectral form that it is more suitable for study of generation higher harmonics in the resonators. The asymptotic solutions are compared with numerical ones.

Solution of model equations

When describing the nonlinear plane standing waves in resonator of a constant radius it is possible to start with the Kuznetsov's model equation for velocity potential ϕ in the second approximation (see e.g. [7])

$$\frac{\partial^2\phi}{\partial t^2} - c_0^2\frac{\partial^2\phi}{\partial x^2} + \frac{\gamma-1}{2c_0^2}\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)^2 = \frac{b}{\rho_0 c_0^2}\frac{\partial^3\phi}{\partial t^3}\ , \quad (1)$$

where x is space coordinate in the direction of the resonator axis, t is time, c_0 is the small signal sound speed, ρ_0 is the ambient density of the fluid, $\gamma = c_p/c_V$ is Poisson's number and c_p, or c_v is the specific heat under constant pressure, or volume, $b = \rho_0\delta$, where δ is the diffusivity of sound (see e.g [8]). In this resonator we can imagine the sound field as a superposition of simple waves propagating in opposite directions which are assumed to not interact in the volume of the resonator and they are coupled only by the conditions on the walls of resonator, see [3]. The next possible simplification is when we neglect the fact that the driving piston is moving and thus the position of the boundary of the resonator is unvarying with the time. This assumption is acceptable for very small amplitude of driving piston. With the above mentioned suppositions we can find the solution of this equation in the following form

$$\phi = \left[\mu\phi_+\left(\mu x, \mu t, \tau_+ = t - \frac{x}{c_0}\right) - \mu\phi_-\left(\mu x, \mu t, \tau_- = t + \frac{x}{c_0}\right)\right]\ , \quad (2)$$

where μ is a small parameter. Substituting the expression (2) into Eq. (1) and neglecting the terms of the order three and higher and supposing that counter-propagating waves do not interact we can get, likewise [6], the following two equations

$$\pm c_0 \frac{\partial v_\pm}{\partial x} + \frac{\partial v_\pm}{\partial t} - \frac{\beta}{c_0} v_\pm \frac{\partial v_\pm}{\partial \tau_\pm} = \frac{b}{2\rho_0 c_0^2} \frac{\partial^2 v_\pm}{\partial \tau_\pm^2} \ . \quad (3)$$

We can write for an acoustic velocity

$$v = v_+ - v_- \ , \quad (4)$$

where v_+ and v_- are solutions of Eqs. (3). The length of the resonator of a constant diameter is labelled by L. It is valid for angular eigenfrequencies ω_n that

$$\omega_n = \frac{n\pi c_0}{L} \ , \quad n = 1, 2, 3, \ldots \ . \quad (5)$$

In the case that we consider the harmonic excitation of the standing waves with the piston at the position $x = L$, we can express the boundary and initial conditions as follows

$$v = (v_+ - v_-)_{x=0} = 0 \ , \quad v_\pm(t = 0) = 0 \ , \quad (6)$$

$$v = (v_+ - v_-)_{x=L} = v_{m1} \sin(\omega t) + v_{m2} \sin(2\omega t + \varphi) \ , \quad (7)$$

where v_{m1} and v_{m2} are acoustic velocity amplitudes of the piston and φ is a phase shift. We assume that a piston vibrates with the angular frequency ω which is equal to $(2n + 1)$-th eigenfrequency of the given resonator, it means that $\omega = \omega_{2n+1}$. This assumption causes that higher harmonic components of an acoustic velocity are in coincidence with eigenfrequencies. Eqs. (3) together with conditions (6) and (7) can be solved by the method of successive approximation, see [6]. On the basis this method we obtain these model equations

$$\frac{\partial \overline{v}_\pm}{\partial t} - \frac{\beta}{c_0} \overline{v}_\pm \frac{\partial \overline{v}_\pm}{\partial \tau_\pm} - \frac{b}{2\rho_0 c_0^2} \frac{\partial^2 \overline{v}_\pm}{\partial \tau_\pm^2} = \quad (8)$$
$$\frac{v_{m1} c_0}{2L} \sin(\omega \tau_\pm) + \frac{v_{m2} c_0}{2L} \sin(2\omega \tau_\pm + \varphi) \ .$$

Eqs. (8) represent the inhomogeneous Burgers equation where

$$\overline{v}_\pm(t, \tau_\pm) = v_\pm(t, \tau_\pm) \pm \frac{v_{m1} x}{2L} \sin(\omega \tau_\pm) \pm \frac{v_{m2} x}{2L} \sin(2\omega \tau_\pm + \varphi) \ . \quad (9)$$

Substituting from Eqs. (9) to Eq. (4) we obtain

$$v(t, x) = \overline{v}_+ - \overline{v}_- - \frac{v_{m1} x}{L} \cos\left(\frac{\omega x}{c_0}\right) \sin(\omega t) \quad (10)$$
$$- \frac{v_{m2} x}{L} \cos\left(\frac{2\omega x}{c_0}\right) \sin(2\omega t + \varphi) \ .$$

It is more suitable to express Eqs. (8) in the dimensionless form

$$\frac{\partial V_\pm}{\partial s} - V_\pm \frac{\partial V_\pm}{\partial y_\pm} - \frac{1}{\Gamma} \frac{\partial^2 V_\pm}{\partial y_\pm^2} = \sin(y_\pm) + p \sin(2y_\pm + \varphi) \ , \quad (11)$$

where

$$s = \frac{t}{t_s} \ , \quad V_\pm = \frac{\overline{v}_\pm}{v_0} \ , \quad y_\pm = \omega \tau_\pm \ , \quad \Gamma = \frac{2\rho_0 c_0 \beta v_0}{b\omega} \ ,$$
$$p = \frac{v_{m2}}{v_{m1}} \ , \quad v_0 = \sqrt{\frac{v_{m1} c_0}{2\pi\beta}} \ , \quad t_s = \frac{c_0}{\beta \omega v_0} \ . \quad (12)$$

Eqs. (11) have the same form for both counter-propagating waves and consequently we can re-mark them because of clarity as

$$\frac{\partial V}{\partial s} - V \frac{\partial V}{\partial y} - \frac{1}{\Gamma} \frac{\partial^2 V}{\partial y^2} = \sin(y) + p \sin(2y + \varphi) \ . \quad (13)$$

Supposing the stationary state ($\partial V / \partial s = 0$) and using the Cole-Hopf transformation (see e.g. [2], [8])

$$V = \frac{2}{\Gamma} \frac{U'}{U} \ , \quad (14)$$

we obtain the following linear differential equation from Eq. (13)

$$U'' + \left[a - \frac{\Gamma^2}{2} \cos(y) - \frac{\Gamma^2 p}{4} \cos(2y + \varphi)\right] U = 0 \ , \quad (15)$$

where comma represents the derivative with respect to y. Eq. (15) is called the Whittaker-Hill equation (see [9], [10]). When p is not very large we can solve Eq. (15) by the asymptotic method, see e.g. [11], [10]. After using the asymptotic method and the transformation (14) we get

$$V(y) \simeq 2 \cos\left(\frac{y}{2}\right) \sqrt{1 + 2p \sin^2\left(\frac{y}{2}\right)} \quad (16)$$
$$\tanh\left[2\Gamma \sin\left(\frac{y}{2}\right) \sqrt{1 + 2p \sin^2\left(\frac{y}{2}\right)}\right] \ .$$

In the case that p is very large it is necessary to modify the argument of the hyperbolic tangent of the solution (16) by omitting the 2

$$V(y) \simeq 2 \cos\left(\frac{y}{2}\right) \sqrt{1 + 2p \sin^2\left(\frac{y}{2}\right)} \quad (17)$$
$$\tanh\left[\Gamma \sin\left(\frac{y}{2}\right) \sqrt{1 + 2p \sin^2\left(\frac{y}{2}\right)}\right] \ .$$

When Γ tends to infinity (an ideal fluid) we get from Eq. (16)

$$V(y) = 2 \cos\left(\frac{y}{2}\right) \operatorname{sign}(y) \sqrt{1 + 2p \sin^2\left(\frac{y}{2}\right)} \ , \quad (18)$$

where sign represents the function signum. The solution (18) was presented in the paper [3]. When $p = 0$ then the solution (16) and (18) takes the form (see [4])

$$V(y) \simeq 2\cos\left(\frac{y}{2}\right)\tanh\left[\Gamma\sin\left(\frac{y}{2}\right)\right] \qquad (19)$$

and

$$V(y) = 2\cos\left(\frac{y}{2}\right)\mathrm{sign}(y)\ . \qquad (20)$$

The expression (19) represents the approximate solution of the inhomogeneous Burgers equation (see e.g. [5])

$$\frac{\partial V}{\partial s} - V\frac{\partial V}{\partial y} - \frac{1}{\Gamma}\frac{\partial^2 V}{\partial y^2} = \sin(y)\ . \qquad (21)$$

We can express the solution (16) by means of the Fourier series

$$V \simeq \sum_{n=1}^{\infty} V_n \sin(ny)\ , \qquad (22)$$

where

$$V_n = \frac{2}{\pi}\int_0^{\pi} 2\cos\left(\frac{y}{2}\right)\sqrt{1+2p\sin^2\left(\frac{y}{2}\right)}$$
$$\tanh\left[2\Gamma\sin\left(\frac{y}{2}\right)\sqrt{1+2p\sin^2\left(\frac{y}{2}\right)}\right]\sin(ny)\mathrm{d}y\ . \qquad (23)$$

Then we can write

$$V_n \simeq \frac{2+\frac{p}{2}}{2\Gamma\sinh\left[\frac{\pi(n+1/2)}{2\Gamma}\right]} + \frac{2+\frac{p}{2}}{2\Gamma\sinh\left[\frac{\pi(n-1/2)}{2\Gamma}\right]} \qquad (24)$$
$$-\frac{p}{4\Gamma\sinh\left[\frac{\pi(n+3/2)}{2\Gamma}\right]} - \frac{p}{4\Gamma\sinh\left[\frac{\pi(n-3/2)}{2\Gamma}\right]}\ .$$

When $\Gamma \to \infty$ (ideal fluid) we obtain from (24) this expression

$$V_n \simeq -\frac{16n(9+2p-4n^2)}{\pi(16n^4-40n^2+9)}\ . \qquad (25)$$

When $p = 0$ then Eq. (25) takes the form

$$V_n = \frac{16n}{\pi(4n^2-1)}\ . \qquad (26)$$

With the help expressions (10) and (22) we can write

$$v(x,t) \simeq -\frac{v_{m1}x}{L}\cos\left(\frac{\omega x}{c_0}\right)\sin(\omega t) +$$
$$\frac{v_{m2}x}{L}\cos\left(\frac{2\omega x}{c_0}\right)\sin(2\omega t)$$
$$-\sqrt{\frac{2v_{m1}c_0}{\pi\beta}}\sum_{n=1}^{\infty}\left[V_n\sin\left(\frac{n\omega x}{c_0}\right)\cos(n\omega t)\right]\ . \qquad (27)$$

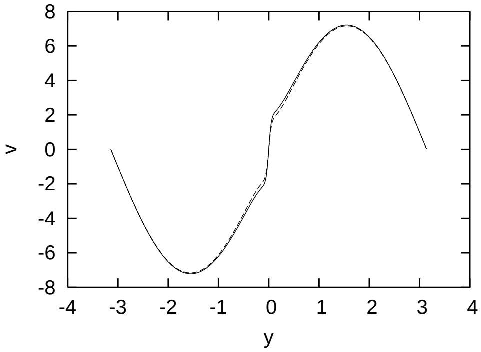

Figure 1: Comparison of the asymptotic solution (solid line) and the numerical solution (dashed line), $\Gamma = 20, p = 25$.

Comparison with numerical results

In this section we deal with comparison between the asymptotic (analytic) and numerical solutions of the inhomogeneous Burgers equation for stationary wave state. The comparison of the asymptotic solution (16) and numerical one is shown in Figs. 1, 2. We can observe that the solutions are in a good agreement. In Fig. 2 it is difficult to distinguish the wave form of the asymptotic and numerical solution. In order to demonstrate the contribution of the new asymptotic solution (16) we compare values of harmonics for various parameters Γ in Fig. 3. For this reason we used the formula (24) with $p = 0$. On the basis of this figure it is obvious that the higher harmonics differ significantly for smaller parameters Γ. Consequently, it is necessary to use for smaller values of Γ the asymptotic solution (16). To illustrate the validity of the spectral solution (24) we made Fig. 4. Here we compare three wave forms. The wave form labelled by 1 represents the asymptotic solution (16) for $p = 0.5$ and $\Gamma = 50$ whereas the wave forms 2 and 3 are obtained from its spectral form (24) for $p = 0$ and $p = 0.5$. We can see that the presented limitation $p \leq 0.5$ enables to get acceptable results. The wave form of (16) for $p = 0$ is not depicted here because differences are not observable.

Acknowledgements

This research has been supported by GAČR grant No. 202/01/1372 and research project No. J04/98:212300016.

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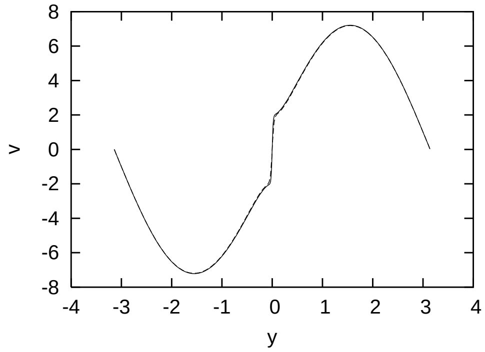

Figure 2: Comparison of the asymptotic solution (solid line) and the numerical solution (dashed line), $\Gamma = 50$, $p = 25$.

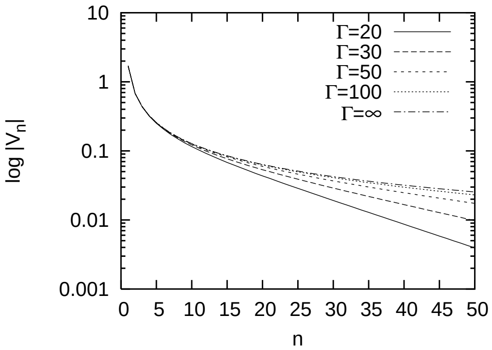

Figure 3: Comparison of the harmonics magnitudes $|V_n|$ for various Γ, and $p = 0$.

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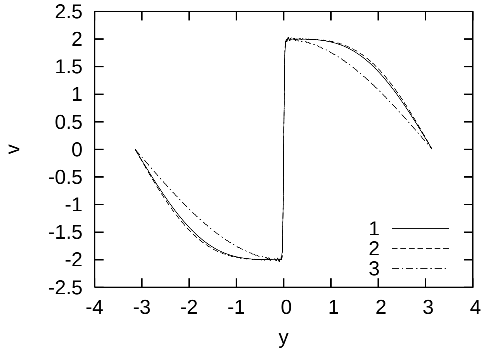

Figure 4: Comparison of the asymptotic solution for $p = 0.5$ (solid line), the spectral form solution for $p = 0.5$ (dashed line), the spectral form solution for $p = 0$ (dashed-dotted line), $\Gamma = 50$.

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