ULTRASONIC NONLINEAR RESONANCE FOR CHARACTERIZATION OF MATERIAL INHOMOGENEITY

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Abstract

Nonlinear elastic material with weak inhomogeneous physical properties and two parallel boundaries is considered. An analytical solution that describes the initial stage of two sine waves simultaneous propagation and interaction in the material is derived. The boundary oscillations caused by wave interaction are analyzed. The phenomenon of nonlinear resonance of boundary oscillation as function of nonlinear variation of amplitude versus excitation frequency is clarified. It turns out that the shift in the nonlinear resonances value is sensitive to the inhomogeneity in the properties of nonlinear elastic material.

Introduction

Nonlinear effects that accompany simultaneous propagation, interaction and reflection of waves in different materials are under intensive investigation due to promising applications of these effects for nondestructive characterization of materials and their states [1, 2]. Effects of resonant wave-wave interaction [3], amplitude amplification by interaction, modulation of amplitudes by different external and internal affects [4] constitute only a short list of phenomena that may be used for practical purposes.

In this study the theoretical basis for two harmonic waves simultaneous propagation in the nonlinear elastic material is presented. The corresponding analytical solution is derived and analyzed numerically. The resonant values of interaction amplitudes as functions of excitation frequency are determined for various physically inhomogeneous nonlinear elastic materials. These resonant values are sensitive to the material properties and they may be used for nondestructive characterization of the materials.

Problem formulation

The one-dimensional motion of the inhomogeneous nonlinear elastic material is described by the equation [4]

$$\begin{aligned} &[\,1 + k_1(X)\,U_{,X}(\,X,t)\,]\;U_{,XX}(\,X,t) \\ &+k_2(X)\,U_{,X}(\,X,t) +\; k_3(X)\,[U_{,X}(\,X,t)]^2 \\ &-k_4(X)\,U_{,tt}(\,X,t) = 0\,. \end{aligned} \tag{1}$$

Here U denotes displacement, X the space coordinate and t the time. The comma and the variables in the indices indicate partial derivations with respect to corresponding variables.

Equation (1) is a nonlinear second order partial differential equation with varialbe in space coefficients. These coefficients $k_i(X)$, $i = 1 \ldots 4$ are functions of variable physical properties of a material, i.e., the density $\rho(X)$, the Lamé coefficients $\lambda(X)$ and $\mu(X)$ and the third order coefficients of elasticity $\nu_1(X)$, $\nu_2(X)$, $\nu_3(X)$ [4]. In the one-dimensional case the five elastic coefficients group together as follows:

$$\begin{aligned} &\alpha(X) = \lambda(X) + 2\mu(X)\,, \\ &\beta(X) = 2\;[\nu_1(X) + \nu_1(2) + \nu_3(X)]\;, \end{aligned} \tag{2}$$

where $\alpha(X)$ is the linear and $\beta(X)$ the nonlinear coefficient of elasticity. Now, the coefficients $k_i(X)$, $i = 1 \ldots 4$ in Eq. (1) may be expressed by formulae

$$\begin{aligned} &k_0(X) = [\alpha(X)]^{-1}\;, \\ &k_1(X) = 3\,[\,1 +\; k_0(X)\,\beta(X)]\;, \\ &k_2(X) = k_0(X)\,\alpha_{,X}(X)\,, \\ &k_3(X) = \frac{3}{2}\,k_0(X)\,[\alpha_{,X}(X) + \beta_{,X}(X)]\;, \\ &k_4(X) = \rho_o(X)\,k_0(X)\,. \end{aligned} \tag{3}$$

It is assumed that the material has two parallel traction-free surfaces at $X = 0$ and $X = L$. Eq. (1) is solved under the initial and boundary conditions

$$\begin{aligned} &U(X,0) = U_{,t}(X,0) = 0\,, \\ &U_{,t}(0,t) =\; \varepsilon\,a_0\;\varphi(t)\;H(t)\,, \\ &U_{,t}(L,t) =\; \varepsilon\,a_L\;\psi(t)\;H(t)\,, \end{aligned} \tag{4}$$

where the arbitrary smooth functions $\varphi(t)$ and $\psi(t)$ determine the initial wave profiles and satisfy conditions $max\,|\varphi(t)| = 1$, $max\,|\psi(t)| = 1$ and $\lim_{t\to 0} U_{,t}(0,t) = \lim_{t\to 0} U_{,t}(L,t) = 0$. Constants εa_0 and εa_L depict the initial wave amplitudes and $H(t)$ is a Heaviside function.

Equation (1) is solved under the initial and boundary conditions (4) making use of the perturbation technique. The analytical solution to Eq. (1) is sought in the form of a series with a small parameter ε:

$$U(X,t) = \sum_{n=1}^{\infty} \varepsilon^n\,U^{(n)}(X,t)\,, \quad 0 < \varepsilon \ll 1. \tag{5}$$

Henceforth, only the first three terms in series (5) are considered.

It is assumed that the inhomogeneity in the density $\rho(X)$, linear elastic coefficient $\alpha(X)$ and nonlinear elastic coefficient $\beta(X)$ of the material is weak and it may be described as a small deviation from the constant value of these properties by expression

$$\gamma(X) = \gamma^{(1)} + \varepsilon\,\gamma^{(2)}(X)\,, \quad \gamma = \rho, \alpha, \beta\,. \tag{6}$$

Here the functions $\gamma^{(2)}(X)$ that describe variable properties of the material are considered as the third order polynomials

$$\begin{aligned}&\gamma^{(2)}(X) = \gamma_{1\xi}X + \gamma_{2\xi}X^2 + \gamma_{3\xi}X^3,\\ &\gamma^{(2)}(X) = \rho^{(2)}(X),\ \alpha^{(2)}(X),\ \beta^{(2)}(X)\,,\\ &\xi = \rho,\ \alpha,\ \beta\,.\end{aligned} \tag{7}$$

Now, the inhomogeneous physical properties of the material are determined by nine constants $\gamma_{i\xi},\ \gamma,\ \xi = \rho, \alpha, \beta,\ i = 1,2,3$.

By introducing Eqs. (6) and (7) into expressions (3) the coefficient $k_0(X)$ may be expanded into the Taylor series

$$k_0(X) = \frac{1}{\alpha^{(1)}}\left[1 - \varepsilon\frac{\alpha^{(2)}(X)}{\alpha^{(1)}} + \left(\varepsilon\frac{\alpha^{(2)}(X)}{\alpha^{(1)}}\right)^2 - \ldots\right]$$

and coefficients $k_i(X),\ i = 1\ldots 4$, after some algebraic modifications have a form

$$\begin{aligned}&k_1(X) = k_1^{(1)} + \varepsilon\,k_1^{(2)}(X) + \varepsilon^2 k_1^{(3)}(X)\,,\\ &k_2(X) = \varepsilon\,k_2^{(2)}(X) + \varepsilon^2 k_2^{(3)}(X)\,,\\ &k_3(X) = \varepsilon\,k_3^{(2)}(X) + \varepsilon^2 k_3^{(3)}(X)\,,\\ &k_4(X) = k_4^{(1)} + \varepsilon\,k_4^{(2)}(X) + \varepsilon^2 k_4^{(3)}(X)\,,\end{aligned} \tag{8}$$

where functions $k_i^{(j)}(X),\ i = 1,\ldots,4,\ j = 2,3$ are polynomials of the 2nd, 3rd, 5th or 6th order.

Finally, by introducing the first three terms in series (5) and expressions (8) into the equation of motion (1) the governing equation yields. This equation is solved by the perturbation procedure, i. e., equating to zero the terms with the same power of ε and neglecting all terms higher than ε^3. The system of equations for determination of the first three terms in series (5) follows

$$O(\varepsilon):\quad U^{(1)}_{,XX}(X,t) - k_4^{(1)}\,U^{(1)}_{,tt}(X,t) = 0\,, \tag{9}$$

$$\begin{aligned}&O(\varepsilon^2):\quad U^{(2)}_{,XX}(X,t) - k_4^{(1)}\,U^{(2)}_{,tt}(X,t)\\ &= -k_1^{(1)}\,U^{(1)}_{,X}(X,t)\,U^{(1)}_{,XX}(X,t) - k_2^{(2)}(X)\,U^{(1)}_{,X}(X,t)\\ &+ k_4^{(2)}(X)\,U^{(1)}_{,tt}(X,t)\,,\end{aligned} \tag{10}$$

$$\begin{aligned}&O(\varepsilon^3):\quad U^{(3)}_{,XX}(X,t) - k_4^{(1)}\,U^{(3)}_{,tt}(X,t) =\\ &-k_2^{(2)}(X)\,U^{(2)}_{,X}(X,t) - k_2^{(3)}(X)\,U^{(1)}_{,X}(X,t)\\ &-k_3^{(2)}(X)\,\left[U^{(1)}_{,X}(X,t)\right]^2 - k_1^{(1)}\,U^{(1)}_{,X}(X,t)U^{(2)}_{,XX}(X,t)\\ &-k_1^{(1)}\,U^{(1)}_{,XX}(X,t)\,U^{(2)}_{,X}(X,t)\\ &-k_1^{(2)}(X)\,U^{(1)}_{,XX}(X,t)\,U^{(1)}_{,X}(X,t)\\ &+k_4^{(2)}(X)\,U^{(2)}_{,tt}(X,t)\\ &+k_4^{(3)}(X)\,U^{(1)}_{,tt}(X,t)\,.\end{aligned} \tag{11}$$

Sine wave interaction

The aim is to investigate simultaneous propagation of two harmonic waves in the physically nonlinear inhomogeneous elastic material. To this end the initial wave profiles are chosen as sine functions

$$\varphi(t) = \psi(t) = \sin(\omega t)\,, \tag{12}$$

where ω denotes the radial frequency.

Wave propagation and interaction is described by the solution (5). The first term in series (5) is the solution to Eq. (9). It is derived under the initial and boundary conditions (4) and (12) and it has a form

$$\begin{aligned}&U^{(1)}_{,t}(X,t) = a_0 H(\xi)\varphi(\xi) + a_L H(\eta)\psi(\eta)\\ &\qquad -a_0 H(\theta)\varphi(\theta) - a_L H(\zeta)\psi(\zeta)\,,\\ &\xi = t - \frac{X}{c}\,,\ \eta = t - \frac{L-X}{c}\,,\ \theta = t - \frac{2L-X}{c}\,,\\ &\zeta = t - \frac{X+L}{c}\,,\end{aligned} \tag{13}$$

where $c = (k_4^{(1)})^{(-1/2)}$ is the linear wave velocity.

The analytical expressions for the second and the third term in solution (5) are derived making use of the symbolic software Maple V. These terms are solutions to Eqs. (10) and (11) under initial and boundary conditions equal to zero. Due to the complexity of these expressions they are not presented here. More information about the analytical expressions for these terms one can find in [4].

Simultaneous propagation, reflection and interaction of two harmonic waves in the material is illustrated on the basis of the results of numerical simulation implemented in consideration of the obtained analytical solution. The material is assumed to be duralumin with the density $\rho^{(1)} = 3000$ kg/m^3 and the coefficients of elasticity $\alpha^{(1)} = 100$ GPa and $\beta^{(1)} = -750$ GPa. The thickness of the material $L = 0.1$ m. Wave process is excited by the values of constants $a_0 = -a_L = -c$ m/s and $\varepsilon = 10^{-4}$.

Wave process is studied in terms of $U_{,X}(X,t)$ that characterizes the stress distribution in the material.

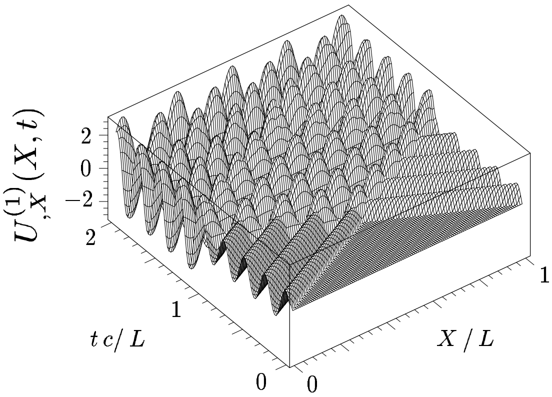

Figure 1: Linear wave profiles superposition evoked by two waves simultaneous propagation.

Function $U_{,X}(X,t)$ is derived from solution (5). The first term $U^{(1)}_{,X}(X,t)$ in solution characterizes stress distribution caused by the linear wave process and subsequent terms $U^{(i)}_{,X}(X,t)$, $i = 2,3$ take the nonlinearity and inhomogeneity of the problem into account.

Linear wave process in terms of $U^{(1)}_{,X}(X,t)$ excited with the frequency $\omega = 1.53447 \cdot 10^6$ rad/s is plotted in Fig. 1. On the boundaries two different intervals may be distinguished: the interval of wave propagation $0 \leq t\,c/L < 1$ and the interval of wave interaction $1 \leq t\,c/L < 2$.

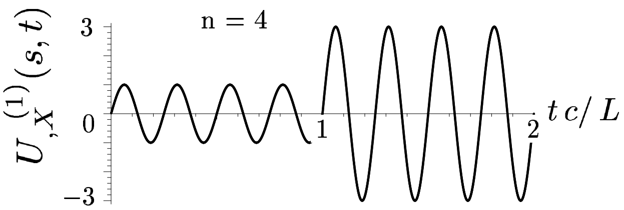

Figure 2: Oscillation on the boundaries of the material.

Amplification of the amplitude of the boundary oscillation in the interval of wave interaction is dependent on the frequency ω. If the frequency is equal to $\omega = 2\,\pi\,n\,c/L$ the amplification is the highest: three times the initial amplitude. This is illustrated in Fig. 2. In the special case when $\omega = 2\,\pi\,(n + 0.5)\,c/L$ there is no amplification in the interval of wave interaction. By other values of ω the amplification is less than in the case $\omega = 2\,\pi\,n\,c/L$.

Main part of the nonlinear boundary oscillation $U^{(2)}_{,X}(X,t)$, plotted in Fig. 3 is characterized by the double frequency i. e., the second harmonic. Here the excitation frequency satisfies the condition $\omega = 6\,\pi\,c/L$ and two different nonlinear elastic materials are

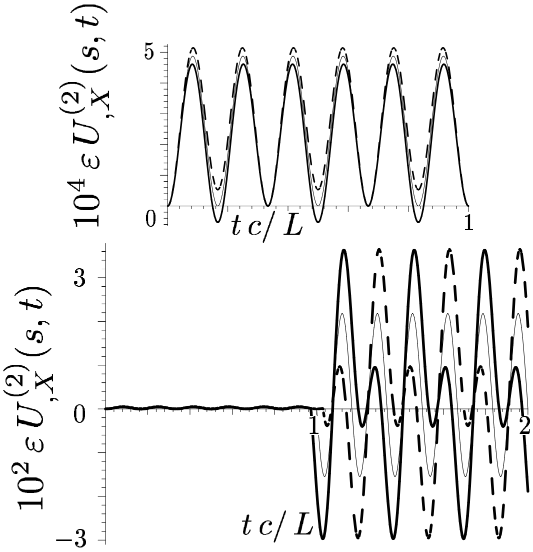

Figure 3: Second order effects of boundary oscillations.

considered. The thin solid line corresponds to the nonlinear oscillation on the boundaries $X = 0$ and $X = L$ of the physically homogeneous material and the bold dashed and bold solid line describe oscillation in inhomogeneous material on boundaries $X = 0$ and $X = L$, respectively. Interesting is that in the interval of wave propagation, magnified in the upper part of Fig. 3 the nonlinear effects are two orders of magnitude smaller that in the interval of wave interaction $1 \leq t\,c/L < 2$. This difference in magnitudes is not dependent on the frequency as it was in the case of linear wave interaction (Fig. 2). Essential is the fact that the oscillation on the boundaries of the homogeneous material has constant amplitude but the inhomogeneity in the physical properties of the material modulates it.

Qualitatively similar are the boundary oscillations described by the term $U^{(3)}_{,X}(X,t)$ except for the phenomenon that these higher order oscillations are modulated already on the boundaries of the homogeneous material.

Material characterization by nonlinear resonance

In this section attention is confined to clarify the possibility to use wave interaction resonance caused by two waves simultaneous propagation in the material for nondestructive material characterization. The resonance (maximum) of the amplitude of the linear interaction (superposition) occurred by the value of the excitation frequency $\omega_l = 8\,\pi\,c/L$ (Fig. 2, n=4). The value of this resonance amplitude A_l depends on the excitation amplitude. The computational simulation is executed by the excitation amplitudes when the conditions $-1.30\,c \le a_0 \le -0.35\,c$ and $a_0 = -a_L$ are satisfied.

Material inhomogeneity modulates the interaction amplitude (Fig. 3). This is the reason why, henceforth, the first positive peak of the interaction amplitude is considered. The value of this peak A is computed on the basis of three first terms in solution (5) and it takes the nonlinearity and material inhomogeneity into account.

Physical properties of the material are assumed to vary linearly along the X-axes and they are described by expression (6) and (7) in the form

$$\gamma(X) = \gamma^{(1)}\,(1 + \delta_{1\xi}(X))\;,\quad \gamma, \xi = \rho, \alpha, \beta\;, \qquad (14)$$

where $\delta_{1\xi}(X) = \varepsilon\gamma_{1\xi}X/\gamma^{(1)}$. The inhomogeneity of material properties is estimated by the value of the parameter of inhomogeneity $\delta_{1\xi}(L) \equiv \delta_{1\xi}, \quad \xi = \rho, \alpha, \beta$.

As an illustration, three different materials with variable linear elastic properties $\alpha(X)$ are considered:

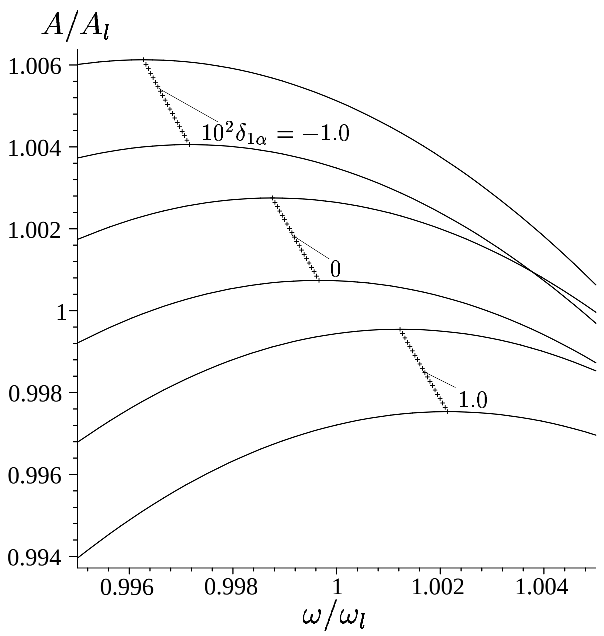

Figure 4: The influence of nonlinearity and physical inhomogeneity on the resonance frequencies.

(i) Elastic coefficient $\alpha(X)$ varies linearly with a negative slope along the X-axes and the maximum deviation at $X = L$ from the basic value $\alpha^{(1)} = 100$ GPa is 1 %, i. e., $10^2\delta_{1\alpha} = -1.0$.

(ii) Elastic coefficient $\alpha(X)$ is constant equal to $\alpha^{(1)} = 100$ GPa (homogeneous material).

(iii) Elastic coefficient $\alpha(X)$ varies linearly with a positive slope along the X-axes ($10^2\delta_{1\alpha} = 1.0$).

The curves relative maximum value of the the first positive peak of the interaction amplitude versus relative excitation frequency are computed for free different materials, mentioned above (Fig. 4).

Conclusions

Analyzes of the curves presented in Fig. 4 enables to draw following conclusions:

(i) Variation in the value of the linear elastic coefficient induces the shift in the value of the resonance frequency.

(ii) Value of the resonance frequency is sensitive to the properties of the material.

(iii) Resonance frequency data may be used for qualitative and quantitative nondestructive characterization of physically inhomogeneous nonlinear elastic material. The last conclusion is supported by the results of similar computational simulations for the material with variable density. In this case the shift of the resonant frequency occurred only in the horizontal direction.

The influence of the variation of the nonlinear elastic coefficient on the resonant frequency is essentially weaker than that of the density and the linear elastic coefficient.

Acknowledgement

The research was supported by the European Science Foundation under the NATEMIS programme and by the Estonian Science Foundation under grant no. 4706.

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