NEURAL COMPUTING OF FINITE ELEMENT STIFFNESS MATRIX

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Abstract

There are cases of major discrepancies between the field predicted by a finite element approximation and the field derived from the exact theory, the so-called "locking effect" being a typical example. This can be tracked to deficiency of interpolation functions. The present paper deals with the approach, which does not need an explicit formulation of the interpolation functions. Instead, it makes use of a representative set of problems resolved in the frameworks of the exact theory and a neural network, which is trained on this set. The elemental stiffness matrix computed by this method is free of the above drawbacks.

Key words: finite element, stiffness matrix, neural network.

1. Introduction

The derivations of the finite element stiffness matrix assume dependence between the nodal values of the field and those inside the element through the explicit interpolation functions and a subsequent resort to either equilibrium equations or energy considerations. Though the interpolation functions are specified so as to comply with approximation theories, they are not always capable of describing in a satisfactory way the field under consideration. In fact, there are cases of major discrepancies between the field predicted by the exact theory and such finite element approximations.

In the context of elasticity, one of the typical phenomena is the "shear locking", which predicts extremely large values of the strain energy for slender finite elements, in a complete disagreement with the exact theory (see, for example, [1, 2]).

This paper deals with an alternative way of computation of the elemental stiffness matrix, which is based on a neural network trained on a representative set of problems resolved in the frameworks of the exact theory. The neural network performs a linear association task and the stiffness matrix of the element follows as the weight matrix of the neural network. This algorithm may also be formulated in terms of the pseudoinverse transform [3, 4].

This approach does not require an explicit formulation of the interpolation functions, enabling one to incorporate in a direct way the results of the exact theory. Consequently, the stiffness matrix obtained is free of the drawbacks brought in by the interpolation techniques.

We first consider a problem of beam to illustrate the method in a simple way and then deal with a plane rectangular elastic element, focusing in particular on the locking phenomenon.

2. Illustrative problem

Consider a problem, which highlights the above neural computing by recovering the well-known result for the stiffness matrix of the elementary beam. The beam element is characterized by four nodal forces $\{q\}=[q_1\ \ q_2\ \ q_3\ \ q_4]^T$ and four nodal displacements $\{w\}=[w_1\ w_2\ w_3\ w_4]^T$ with the stiffness matrix [K] defined by

$$\{q\}=[K]\ \{w\} \tag{1}$$

We consider {w} and {q} in (1) as a linear neural network input and output, respectively, and formulate the following problems of the beam loading: 1) cantilever subjected to a concentrated force at its left end, 2) cantilever subjected to a concentrated force at its right end, 3) hinged beam under a uniform distributed load, 4) beam built-in at its right end and hinged at its right end under a uniform distributed load.

Under proper normalization of the external forces and the beam length, the exact solutions for these problems provide the following "input-output" pairs:

$$\begin{aligned}
&\{w_1\}=[1/3\ -1/2\ 0\ 0]_T\\
&\{q_1\}=[1\ 0\ -1\ 1]^T\\
&\{w_2\}=[0\ 0\ -1/3\ -1/2]^T\\
&\{q_2\}=[1\ 1\ -1\ 0]^T\\
&\{w_3\}=[0\ 1/24\ 0\ -1/24]^T\\
&\{q_3\}=[0\ 1/12\ 0\ -1/12]^T\\
&\{w_4\}=[0\ 0\ 0\ -1/48]^T\\
&\{q_4\}=[-1/8\ -1/24\ 1/8\ -1/12]^T,
\end{aligned} \tag{2}$$

respectively.

The rigid body motion of the element must cause zero nodal forces. Thus, additional training patterns may be stated as

$$\begin{aligned}
&\{w_5\}=[1\ 0\ 1\ 0]^T\\
&\{q_5\}=[0\ 0\ 0\ 0]^T\\
&\{w_6\}=[-1\ 1\ 0\ 1]^T
\end{aligned} \tag{3}$$

$\{q_6\}=[0\ 0\ 0\ 0]^T$

which describe the translation and rotation of the element as a rigid body, respectively.

Formulating, as the neural network paradigm suggests, the total input [P] and total output [T] as

$$[P]=[\{w_1\}\ \{w_2\}\ \{w_3\}\ \{w_4\}\ \{w_5\}\ \{w_6\}]$$
$$[T]=[\{q_1\}\ \{q_2\}\ \{q_3\}\ \{q_4\}\ \{q_5\}\ \{q_6\}] \tag{4}$$

we get the basic relation as

$$[T]=[W][P]+[b] \tag{5}$$

where [W] is the unknown weight matrix and [b] is the unknown bias matrix. If the initial forces are absent, then [b]=0, and [W] is equivalent to the stiffness matrix [K]. In general, the best solution to (5) (in the least squares sense) is given by

$$[W]=[T][P]^* \quad ([b]=0) \tag{6}$$

where [P]* is the pseudoinverse of [P] (for a theory of pseudoinverse transform the reader is referred to [3, 4]).

Computing (6), we get the weight matrix [W] as follows:

$$[W]=\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \tag{7}$$

which precisely coincides with the stiffness matrix [K] of the beam element.

The iterative Widrow-Hoff algorithm for training the ADALINE neural network [3] is a useful alternative to (6), if the computation of [P]* is difficult. Indeed, in the case at hand this algorithm also provides (7) as the weight matrix.

It should be noted that applications of the least squares method (LMS) to the finite element techniques are well-known (see, for example, [5]). These applications deal with solutions to the governing equations by the LSM in the frameworks of the classic formulation, which involves interpolation functions, and are completely different from the present approach, assuming a representative set of training patterns and a neural network.

3. Plane elasticity

The above problem, being one-dimensional, does not display "locking". To this end, consider plane elastic elements [1, 2]. The strain matrix $\{\varepsilon\}$ and the nodal displacement matrix {w} are related by

$$\{\varepsilon\}=[B]\{w\} \tag{8}$$

with [B] being the strain-displacement matrix. In terms of [B] the stiffness matrix [K] is given by

$$[K]=\iint_S [B]^T[H][B]dxdy \tag{9}$$

where S is the elemental area and [H] the matrix of elastic constants, which for anisotropic materials under the plane stress takes the form

$$[H]=\begin{bmatrix} n & n\nu_2 & 0 \\ n\nu_2 & 1 & 0 \\ 0 & 0 & m(1-n\nu_2^2) \end{bmatrix} E_2/(1-n\nu_2^2) \tag{10}$$

Here E denotes Young's modulus and ν Poisson's ratio, $n=E_1/E_2$, $m=G_1/G_2$, and subscripts 1 and 2 denote the in-plane response of the strata and that in a direction normal to the strata, respectively.

Under the assumption of the constant strain within the element, it is possible to replace (8) by the relation

$$\{\varepsilon\}_o=[B]_o\{w\} \tag{11}$$

where $\{\varepsilon\}_o$ and $[B]_o$ are the averaged values and (9) by the relation

$$[K]= [B]_o^T[H]\,[B]_oS \tag{12}$$

For a rectangular element, which has eight degrees of freedom, the above matrices $\{\varepsilon\}$, [B] and {w} are (3 x 1), (3 x 8) and (8 x 1), respectively.

Consider the element of the size 2a and 2b and denote the displacement along x_i by u_i. We include the following typical cases of loading in the training set: 1) uniform tension in the two directions, 2) uniform shear, 3) bending described by the boundary condition $u_1(x_1=b)=-\,u_1(x_1=-b)$, 4) bending described by the boundary condition $u_1(x_1=0,\ x_2=0)=0$, $u_2(x_1=0)=0$, 5) tension in one direction and compression in the other one, 6) the same but the directions are reversed. The problems are amenable to the exact solution, for example, with the help of Airy's function. The solutions are

$\{\varepsilon\}_1=[5/4\ \text{-}5/4\ 0]^T$
$\{w\}_1=[\text{-}y\ z\ y\ z\ y\ \text{–}z\ \text{–}y\ \text{–}z]^T$, y=5/8, z=1/8

$\{\varepsilon\}_2=[0\ 0\ \text{-}5/2]^T$
$\{w\}_2=[y\ z\ y\ \text{–}z\ \text{–}y\ \text{–}z\ \text{–}y\ z]^T$, y=1/8, z=5/8

$\{\varepsilon\}_3=[6x_2\ \text{-}3x_2/2\ 0]^T$
$\{w\}_3=[y\ \text{–}z\ \text{–}y\ \text{–}z\ y\ \text{–}z\ \text{–}y\ \text{-}z]^T$, y=3/10, z=303/400

(13)

$\{\varepsilon\}_4=[x_2\ 0\ 0]^T$
$\{w\}_4=[y\ \text{–}z\ \text{–}y\ \text{–}z\ y\ \text{–}z\ \text{–}y\ \text{-}z]^T$, y=1/20, z=1/8

$\{\varepsilon\}_5=[\text{-}3/2\ 39/4\ 0]^T$
$\{w\}_5=[y\ \text{–}z\ \text{–}y\ \text{–}z\ \text{–}y\ z\ y\ z]^T$, y=3/4, z=39/40

$\{\varepsilon\}_6=[\text{-}39/4\ 3/2\ 0]^T$
$\{w\}_6=[q\ \text{–}g\ \text{–}q\ \text{–}g\ \text{–}q\ g\ q\ g]^T$, q=39/8,g=3/20

Rigid body motions correspond to vanishing strains, which provides

$\{\varepsilon\}_7=[0\ 0\ 0]^T$
$\{w\}_7=[1\ 0\ 1\ 0\ 1\ 0\ 1\ 0]^T$

(14)

$\{\varepsilon\}_8=[0\ 0\ 0]^T$
$\{w\}_8=[0\ 1\ 0\ 1\ 0\ 1\ 0\ 1]^T$

After integrating the strains over the quarter of the element to arrive at the averaged values, the above relations state eight "input-output" training pairs

$$[P]=[\{w_1\}\ \{w_2\}\ \{w_3\}\ \{w_4\}\ \{w_5\}\ \{w_6\}\ \{w_7\}\ \{w_8\}]$$
$$[T]=[\{\varepsilon_1\}\ \{\varepsilon_2\}\ \{\varepsilon_3\}\ \{\varepsilon_4\}\ \{\varepsilon_5\}\ \{\varepsilon_6\}\ \{\varepsilon_7\}\ \{\varepsilon_8\}] \qquad (15)$$

This enables one to find the $[B]_o$-matrix with the help of (6) as the weight matrix of the neural network which performs the best correspondence between [P] and [T] as given by (15). Then the stiffness matrix [K] follows form (12). It should be noted that the versatility of the finite element depends on the training set in a direct way. Generally, the set should be as comprehensive as possible.

In what follows we consider a material with n=m=1, $E_2/(1-n\nu_2^2)=1$ (isotropic case) and $\nu=1/4$. Calculations show that for the case $a\approx b$, both, the classic approach and the present one, provide values of the elemental strain energy in a reasonable agreement with the exact results. For a slender element the situation is completely different. Table 1 given in the Appendix shows that the classic element fails in the cases 3 and 4, exhibiting a profound locking, while the finite element derived by the present approach is free of this effect.

4. Conclusions

The presented approach does not require an explicit formulation of the interpolation functions and enables one to incorporate in a direct way the relevant results of the exact theory. Consequently, the stiffness matrix obtained is free of the drawbacks brought in by the interpolation techniques. Though a comprehensive training set would lead to a more versatile finite element, by formulating a specialized training set one may arrive at a stiffness matrix, particularly suitable for the problem at hand.

References

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Appendix

Table 1. Strain energy of the element, 2a=1/5, 2b=1.

Pattern	Classic	Present	Exact
1	0.25	0.25	0.25
2	0.25	0.25	0.25
3	0.130	0.009	0.012
4	0.0037	0.0003	0.0004
5	9.6	9.6	9.6
6	9.6	9.6	9.6