RESONANT EXCITATION OF SHOCK-FREE, HIGH-AMPLITUDE OSCILLATIONS OF A GAS COLUMN IN A TUBE

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Abstract

This paper introduces a novel method to generate shock-free, high-amplitude oscillations of a gas column confined in a closed tube by driving it resonantly. Suppression of the shock is achieved by connecting an array of Helmholtz resonators along the tube. The array yields wave dispersion which makes the tube of uniform cross-section *dissonant*. Formulation of the problem is given within the framework of one-dimensional model taking account of both dissipative effects due to the boundary layer and the jet loss in the throat of the resonators. Making use of the method of multiple scales, the cubic nonlinear theory is developed to derive the amplitude equation. The frequency response is obtained and compared with experiments. Discussions are given on restrictions of the theory and physical mechanisms to achieve shock-free oscillations.

Introduction

Resonant excitation of a gas column confined in a tube has so far been studied by many authors (see [1] for review). A gas column of uniform cross-section is commonly driven at its lowest resonant frequency by a reciprocating piston placed at one end of the tube. As the displacement amplitude of the piston is increased, the oscillations are characterised by emergence of a shock. Once the shock emerges, it hinders increase in the peak pressure against increase in the displacement amplitude, though there seems to be no upper bound in the peak pressure.

While emergence of the shock appears to be natural and robust, it has recently been revealed that the shock can be suppressed to achieve high-amplitude oscillations. This is understood by considering mechanisms behind emergence of the shock. In the tube of uniform cross-section, higher resonance frequencies (eigenfrequencies) are ordered as multiples of the lowest one, and the tube length corresponds to multiples of a half wavelength. Such a tube is called *consonant* and otherwise *dissonant* [2]. When the gas column is driven at the lowest frequency, as is usually the case, higher harmonics of excitation generated by nonlinearity coincide with higher modes in frequencies and wavenumbers as well. Thus energy input in the lowest mode is transferred to the one in higher modes so that the shock emerges.

To suppress the shock, it is required to block this cascade process by any means. Rudenko *et al.* [3] and Andreev *et al.* [4] devised to damp the second harmonics externally, while Gusev *et al.* [5] abandoned a monochromatic excitation to control a driver actively by adding its second harmonic component with a suitable phase difference. But it is uncertain whether or not these methods can be applied to achieve shock-free oscillations of peak pressure, say 10% of the ambient pressure as a bench mark. At present, there are two methods available to achieve this level of oscillations or even higher.

One method is to make the tube dissonant by changing the cross-section of the tube. Lawrenson *et al.* [6] and Ilinskii *et al.* [7] designed carefully a tube of non-uniform cross-section and attained extremely high amplitude by oscillating the tube on a shaker. Another method is to exploit wave dispersion in propagation in the tube [8,9]. By connecting an array of Helmholtz resonators (see Fig. 1), the dispersion can be introduced so that the phase speed may be made dependent on the frequency. Then the tube of uniform cross-section is made dissonant automatically.

This paper reviews the latter method to show how the shock-free oscillations are achieved and to obtain a frequency response. Formulation of the problem and derivation of the amplitude equation are outlined and the frequency response is compared against the experiments. Discrepancy between the theory and experiments is examined from a viewpoint of higher-harmonic resonance and evanescence, and some discussions are also given on difference of the two methods described above.

Formulation of the problem

Physical model and basic equations

Formulation is made by considering a physical model relevant to the problem. For the details, reference should be made to the paper [9].

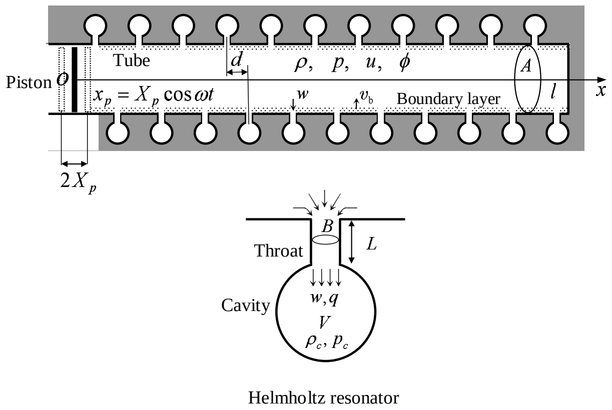

Figure 1: Tube with an array of Helmholtz resonators and a Helmholtz resonator.

The tube with the array of resonators is characterized by the *size parameter* of the array κ and the natural angular frequency of the resonator ω_0 defined, respectively, as

$$\kappa = \frac{V}{Ad} \quad \text{and} \quad \omega_0 = \sqrt{\frac{a_0^2 B}{L_e V}}, \tag{1}$$

where A, d and V denote, respectively, the cross-sectional area of the tube, the axial spacing between neighbouring resonators, and the volume of the cavity, while a_0, B, and L_e denote, respectively, the sound speed, the cross-sectional area of the throat of the resonator, and its effective length, which will be specified later.

The Reynolds number is usually very high so dissipative effects due to viscosity and heat conduction are confined in a thin boundary layer adjacent to the tube wall. Thus almost one-dimensional motion is assumed over the cross-section of the main-flow region outside of the boundary layer. The equations of continuity and of motion are averaged over the cross-section and given as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = \frac{1}{A}\oint \rho v_n \mathrm{d}s, \tag{2}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{3}$$

where ρ, u, and p denote, respectively, the density of the gas, the axial velocity, the pressure averaged over the main-flow region, while v_n denotes the small velocity component inward normal to the boundary of the cross-section of the main flow, ds being the line element along the boundary, and the cross-sectional area of the main-flow region being regarded substantially as A because the boundary layer is very thin, and those variables depend on the axial coordinate x and the time t only. Since no dissipative effects are considered, the adiabatic relation holds: $p/p_0 = (\rho/\rho_0)^\gamma$, the subscript 0 implying the values in equilibrium, and γ the ratio of specific heats.

For the gas in the cavity of the resonator, no motions are assumed and only the equation of continuity is required as follows:

$$V\frac{\partial \rho_c}{\partial t} = Bq, \tag{4}$$

where ρ_c and q ($= \rho w$) denote, respectively, the mean density of the gas in the cavity and the mass flux density flown into the cavity, w being the velocity flown into the cavity and assumed uniform over the cross-section of the throat except for the boundary layer on the throat wall. Denoting the mean pressure in the cavity by p_c, it is related to ρ_c through the adiabatic relation: $p_c/p_0 = (\rho_c/\rho_0)^\gamma$.

The gas in the throat is regarded as being incompressible and the equation of motion is given by taking account of the wall friction on the throat wall and the jet loss as follows:

$$L_e\frac{\partial q}{\partial t} = p' - p'_c - \frac{2L'\sqrt{\nu}}{r}\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}(\rho_0 w) - \rho w|w|, \tag{5}$$

with the excess pressures $p' = p - p_0$ and $p'_c = p_c - p_0$ where L_e is given by $L + 2 \times 0.82r$ taking account of the so-called end corrections, L and r being the throat length and radius, respectively, and the throat is also lengthened to L' ($= L + 2r$) to include the viscous end corrections, ν being the kinematic viscosity of the gas. By the assumption of the incompressible flow with the density ρ and the velocity w, the term last but one on the right-hand side accounts for the linear wall friction due to the boundary layer, while the last term accounts for the jet loss due to asymmetry in the flow pattern on the suction and ejection sides of the throat (see Fig.1). The derivative of order j ($= -1/2, 1/2$ or $3/2$) of a function $f(x,t)$ is defined as follow:

$$\frac{\partial^j f}{\partial t^j} \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{1}{\sqrt{t-\tau}}\frac{\partial^{j+\frac{1}{2}}}{\partial \tau^{j+\frac{1}{2}}}f(x,\tau)\mathrm{d}\tau. \tag{6}$$

Effects of the boundary layer on the tube wall and the array of resonators are taken into account through the right-hand side of (2). Where the tube wall is present, v_n is taken as the velocity at the edge of the boundary layer v_b given by

$$v_b = C\sqrt{\nu}\frac{\partial^{-\frac{1}{2}}}{\partial t^{-\frac{1}{2}}}\left(\frac{\partial u}{\partial x}\right), \tag{7}$$

with $C = 1 + (\gamma - 1)/\sqrt{Pr}$, Pr being the Prandtl number, while where the tube opens to the throat of the resonator, v_n is taken as $-w$. Thus the right-hand side of (2) is evaluated as

$$\frac{1}{A}\oint \rho v_n = \frac{2C\rho\sqrt{\nu}}{R^*}\frac{\partial^{-\frac{1}{2}}}{\partial t^{-\frac{1}{2}}}\left(\frac{\partial u}{\partial x}\right) - \kappa\frac{\partial \rho_c}{\partial t}, \qquad (8)$$

where the continuum approximation for distribution of the resonators has been made so that ρ_c, p_c and w are assumed to depend not only on t but also on x, and R^* is the reduced radius of the tube given by $R/(1-BR/2Ad)$ for a tube of radius R.

Hence the six unknowns ρ, u, p, ρ_c, p_c and w are governed by the six equations (2) with (8), (3), (4), (5) and two adiabatic relations. For the gas column of length l in equilibrium, the boundary conditions are imposed on the piston surface at $x = x_p(t) = X_p\cos(\omega t)$ and on the flat plate at the closed end $x = l$, the origin of the x axis taken at the mean position of the piston surface (see Fig.1). Thus they are given as follows:

$$u = \frac{\mathrm{d}x_p}{\mathrm{d}t} = \frac{\mathrm{i}}{2}\omega X_p \mathrm{e}^{\mathrm{i}\omega t} + c.c. \quad \text{at} \quad x = x_p, \qquad (9)$$

and

$$u = 0 \quad \text{at} \quad x = l, \qquad (10)$$

where the complex amplitude of the displacement X_p and the angular frequency ω are taken real and positive, *c.c.* designating the complex conjugate to the preceding term. Since the stationary oscillations are concerned here, no initial conditions are imposed in this problem.

Summary of equations normalized and parameters

To simplify the equations, the velocity potential ϕ is introduced through $u = \partial\phi/\partial x$ and all variables are made dimensionless by the following replacement:

$$[x,\ t,\ \phi,\ \rho,\ \rho_c,\ p',\ p_c'] \rightarrow$$
$$[lx,\ lt/a_0,\ lu_0\phi,\ \rho_0\rho,\ \rho_0\rho_c,\ \rho_0 a_0 u_0 p',\ \rho_0 a_0 u_0 p_c'],$$

where u_0 denotes a typical (maximum) speed of the gas. In this process, an acoustic Mach number ε and a piston Mach number ε_p are introduced as

$$\varepsilon \equiv \frac{u_0}{a_0} \ll 1 \quad \text{and} \quad \varepsilon_p \equiv \frac{\omega X_p}{a_0} \ll 1. \qquad (11)$$

In the tube with no array, the gas column resonates in the lowest mode when a half-wavelength $\pi a_0/\omega$ coincides with the tube length. Using this frequency, ω is made dimensionless in terms of σ as

$$\omega = \frac{\pi a_0}{l}\sigma. \qquad (12)$$

Elimination of ρ between (2) and (3) with the adiabatic relation leads to the equation for ϕ as follows:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2}$$
$$= \frac{\kappa a^2}{\varepsilon\rho}\frac{\partial\rho_c}{\partial t} - \varepsilon\left[\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)^2 + (\gamma-1)\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial x^2}\right]$$
$$-\varepsilon^2\frac{(\gamma+1)}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2} - \delta a^2\frac{\partial^{-\frac{1}{2}}}{\partial t^{-\frac{1}{2}}}\left(\frac{\partial^2\phi}{\partial x^2}\right). \quad (13)$$

Here a is the local sound speed a $(= \sqrt{\mathrm{d}p/\mathrm{d}\rho})$ and given by

$$a^2 = 1 - \varepsilon(\gamma-1)\left[\frac{\partial\phi}{\partial t} + \frac{\varepsilon}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]. \qquad (14)$$

Elimination q between (4) and (5) and use of the adiabatic relation lead to the equation for p_c' as

$$\frac{\partial^2 p_c'}{\partial t^2} + \delta_r\frac{\partial^{\frac{3}{2}}p_c'}{\partial t^{\frac{3}{2}}} + (\pi\sigma_0)^2 p_c' = (\pi\sigma_0)^2 p'$$
$$+\varepsilon\frac{(\gamma-1)}{2}\frac{\partial^2 p_c'^2}{\partial t^2} - \varepsilon^2\frac{(\gamma-1)(2\gamma-1)}{6}\frac{\partial^2 p_c'^3}{\partial t^2}$$
$$-\varepsilon\delta_J\left|\frac{\partial p_c'}{\partial t}\right|\frac{\partial p_c'}{\partial t} + ..., \qquad (15)$$

with p' given by

$$p' = -\frac{\partial\phi}{\partial t} + \frac{\varepsilon}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right]$$
$$+\frac{\varepsilon^2}{2}\left[\frac{(\gamma-2)}{3}\left(\frac{\partial\phi}{\partial t}\right)^3 + \frac{\partial\phi}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)^2\right] + ..., \quad (16)$$

where $\sigma_0 = l\omega_0/\pi a_0$ and the parameters δ, δ_r and δ_J are defined as follows:

$$\delta = \frac{2C}{R^*}\sqrt{\frac{\nu l}{a_0}},\ \delta_r = \frac{2}{r^*}\sqrt{\frac{\nu l}{a_0}},\ \delta_J = \frac{V}{BL_e}, \qquad (17)$$

and $r^* = L_e r/L'$.

Finally the boundary conditions are normalized as

$$\frac{\partial\phi}{\partial x} = \mathrm{i}\frac{\varepsilon_p}{2\varepsilon}\mathrm{e}^{\mathrm{i}\pi\sigma t} + c.c. \ \text{at} \ x = \frac{c}{2}\mathrm{e}^{\mathrm{i}\pi\sigma t} + c.c., \quad (18)$$

with $\varepsilon_p = \pi\sigma c$ and $c = X_p/l$ $(\ll 1)$ and

$$\frac{\partial\phi}{\partial x} = 0 \ \text{at} \ x = 1. \qquad (19)$$

Resonance and evanescence

In order to show that the tube with the array is dissonant, we examine linear, lossless oscillations by taking the limit as δ, δ_r, ε_p and $\varepsilon \to 0$ with $\varepsilon_p/\varepsilon$ fixed. Then the basic equations become

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \kappa\frac{\partial p_c'}{\partial t}, \tag{20}$$

$$\frac{\partial^2 p_c'}{\partial t^2} + (\pi\sigma_0)^2 p_c' = (\pi\sigma_0)^2 p', \tag{21}$$

$$p' = -\frac{\partial\phi}{\partial t}, \tag{22}$$

with the boundary conditions

$$\frac{\partial\phi}{\partial x} = \mathrm{i}\frac{\varepsilon_p}{2\varepsilon}\mathrm{e}^{\mathrm{i}\pi\sigma t} + c.c. \quad \text{at} \quad x = 0, \tag{23}$$

$$\frac{\partial\phi}{\partial x} = 0 \quad \text{at} \quad x = 1. \tag{24}$$

Eliminating p' and p_c' in (20)-(22), it follows that

$$\left[\frac{\partial^2}{\partial t^2} + (\pi\sigma_0)^2\right]\left(\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2}\right) + \kappa(\pi\sigma_0)^2\frac{\partial^2\phi}{\partial t^2} = 0. \tag{25}$$

The solutions are easily obtained as

$$\begin{bmatrix}\phi\\ p'\\ p_c'\end{bmatrix} = \begin{bmatrix}\mathrm{i}/\pi\sigma\\ 1\\ s\end{bmatrix}\frac{\pi\sigma\cos[k(x-1)]}{k\sin k}\frac{\varepsilon_p}{2\varepsilon}\mathrm{e}^{\mathrm{i}\pi\sigma t} + c.c., \tag{26}$$

with

$$k^2 = (\pi\sigma)^2\left(1 + \frac{\kappa\sigma_0^2}{\sigma_0^2 - \sigma^2}\right), \tag{27}$$

and $s = \sigma_0^2/(\sigma_0^2 - \sigma^2)$. This relation gives the dispersion relation.

The solution (26) suggests that the resonance occurs when $k = m\pi \geq 0$ $(m = 0, 1, 2, ...)$. For $k = 0$, i.e. at $\sigma = \sqrt{1+\kappa}\sigma_0$, the gas column oscillates in unison. This is a new type of resonance supported by the array of resonators. Usual resonance corresponds to the case for $k > 0$. But there appear a pair of resonance frequencies $\sigma_m^\pm$ (> 0) for each m, which are given by

$$(\sigma_m^\pm)^2 = \frac{1}{2}\left\{m^2 + (1+\kappa)\sigma_0^2 \pm\sqrt{[m^2 + (1+\kappa)\sigma_0^2]^2 - 4m^2\sigma_0^2}\right\}, \tag{28}$$

with the signs ordered vertically. Here m specifies the spatial mode of oscillations. Figure 2 shows the dispersion relation (27) in the case with $\kappa = 0.2$ and $\sigma_0 = 2.5$ where the resonance frequencies $\sigma_m^\pm$ $(m = 1, 2, ..., 6)$ are marked by the dots. It is seen that $\sigma_m^- < \sigma_0 < \sqrt{1+\kappa}\sigma_0 < \sigma_m^+$ and no real σ exists between σ_0 and $\sqrt{1+\kappa}\sigma_0$. In this range, the evanescence occurs and no oscillations can be transmitted from the piston. That the tube is dissonant is easily found since the resonance frequencies are off the straight line $\sigma = \sigma_1^- k/\pi$.

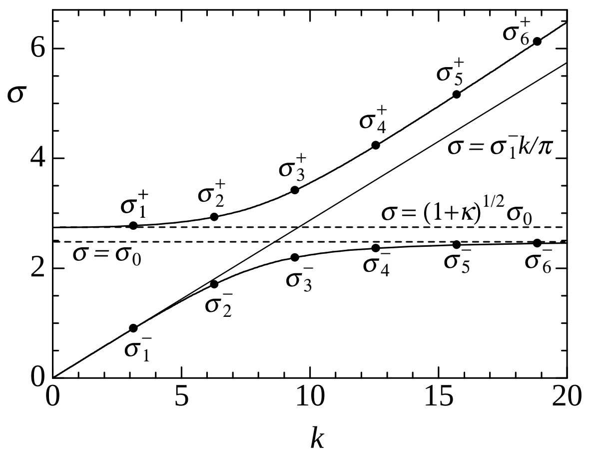

Figure 2: Dispersion relation (27) and resonant frequencies $\sigma_m^\pm$ $(m = 1, 2, ..., 6)$.

Nonlinear theory

Among many parameters introduced by normalization, the magnitude of ε is left unspecified. It is estimated to of $\varepsilon_p/k\sin k$ by the linear lossless solution because the left-hand side of (26) are regarded as quantities of order unity. When σ is close to the lowest resonance frequency σ_1^- $(\equiv \sigma_1)$ with $k \approx \pi$, $\sin k$ becomes small of $-(\mathrm{d}k/\mathrm{d}\sigma)|_{\sigma=\sigma_1}\Delta\sigma$, where $\Delta\sigma$ $(= \sigma - \sigma_1)$ measures the detuning from σ_1 and $|\Delta\sigma| \ll 1$. Thus ε is estimated as

$$\varepsilon \approx \frac{\varepsilon_p}{|\Delta\sigma|}. \tag{29}$$

The experiments suggest that the magnitude of ε_p is of order 10^{-3}, while the one of ε is of order 10^{-1} and $\Delta\sigma$ is of order 10^{-2}. In view of these, the following analysis assumes that ε_p is of ε^3 and $\Delta\sigma$ is of ε^2. Furthermore δ and δ_r are estimated to be of ε^2. But note that δ_J is not small.

The solution for stationary oscillations of the gas column is sought in the form of the asymptotic expansion in terms of ε as follows:

$$\begin{bmatrix}\phi\\ p'\\ p_c'\end{bmatrix} = \begin{bmatrix}\phi^{(0)}\\ f^{(0)}\\ g^{(0)}\end{bmatrix} + \varepsilon\begin{bmatrix}\phi^{(1)}\\ f^{(1)}\\ g^{(1)}\end{bmatrix} + \varepsilon^2\begin{bmatrix}\phi^{(2)}\\ f^{(2)}\\ g^{(2)}\end{bmatrix} + ... \tag{30}$$

The lowest-order (zero-order) solution is taken as

$$\begin{bmatrix} \phi^{(0)} \\ f^{(0)} \\ g^{(0)} \end{bmatrix} = \begin{bmatrix} \mathrm{i}/\pi\sigma_1 \\ 1 \\ s_1 \end{bmatrix} \cos[\pi(x-1)]\alpha \mathrm{e}^{\mathrm{i}\pi\sigma_1 t_0} + c.c., \tag{31}$$

with $s_1 = \sigma_0^2/(\sigma_0^2 - \sigma_1^2)$ ($\sigma_1 \neq \sigma_0$) and $\pi^2\sigma_1^2(1 + \kappa s_1) = \pi^2$. Here the detuning $\Delta\sigma$ is taken into account by allowing a complex amplitude α to depend on a slow time-scale t_2 defined by $\varepsilon^2 t$, while the phase is varying rapidly in t_0 ($= t$). The behaviour of α in t_2 is to be determined so that the expansion (30) may become uniformly valid. To this end, the method of multiple (two) scales is employed and the amplitude equation is derived. For the details, reference should be made to the paper [9].

Here we note some findings in the process of carrying out the expansion. In the first-order solution, a steady pressure distribution appears along the tube, giving in $f^{(1)}$ and $g^{(1)}$ the component

$$\frac{1}{2}\left\{1 - \frac{1}{\sigma_1^2} + \left(1 + \frac{1}{\sigma_1^2}\right)\cos[2\pi(x-1)]\right\}|\alpha|^2. \tag{32}$$

This pressure becomes positive at both ends of the tube and negative in the middle. But no steady streaming is not covered in the present framework because the velocity is averaged over the cross-section of the tube. It is to be remarked that although the jet loss appears in the first-order problem, it is taken in the second-order problem because it is multiplied by κ, which is comparable with ε.

Carrying out the expansion up to the second-order problem, the amplitude equation for α is derived from the boundary conditions as follows:

$$\mathrm{i}\mu\frac{\partial\alpha}{\partial t_2} + \frac{S}{\varepsilon^2}\alpha + \mathrm{i}\frac{D_0}{\varepsilon}|\alpha|\alpha + Q_0|\alpha|^2\alpha = \sigma_1\frac{\varepsilon_{p1}}{\varepsilon^3}\mathrm{e}^{\mathrm{i}\pi\sigma' t_2}, \tag{33}$$

with $\varepsilon_{p1} = \pi\sigma_1 c$ and $\sigma' = \Delta\sigma/\varepsilon^2$ where μ, S ($= S_{re} + \mathrm{i}S_{im}$), and D_0 are given by

$$\mu = 2 + 2\kappa s_1\left(1 + \frac{s_1}{\Omega}\right), \tag{34}$$

$$S_{re} = -S_{im} = -\sqrt{\frac{\pi\sigma_1}{2}}\left(\frac{\delta}{\sigma_1^2} + \frac{\kappa\delta_r s_1^2}{\Omega}\right), \tag{35}$$

$$D_0 = \frac{128\kappa\delta_J\sigma_1^3 s_1^3}{9\pi\sigma_0^2}, \tag{36}$$

with $\Omega = \sigma_0^2/\sigma_1^2$. The real coefficient Q_0 is too complicated to be given explicitly here and is shown graphically in Figure 4 where Q ($= Q_0/4\gamma^2$) is drawn against σ_0 with κ fixed at 0.1, 0.2 and 0.3. The value of Q may be positive or negative but it diverges at $\sigma_0 = 0$ and at $\sigma_0 = \sqrt{4(3-\kappa)/3(1+\kappa)^2}$ where $2\sigma_1$ hits $\sqrt{1+\kappa}\sigma_0$ for resonance with $k = 0$. Although the form of Q is very complicated, it is well approximated for $\Omega \gg 1$ as

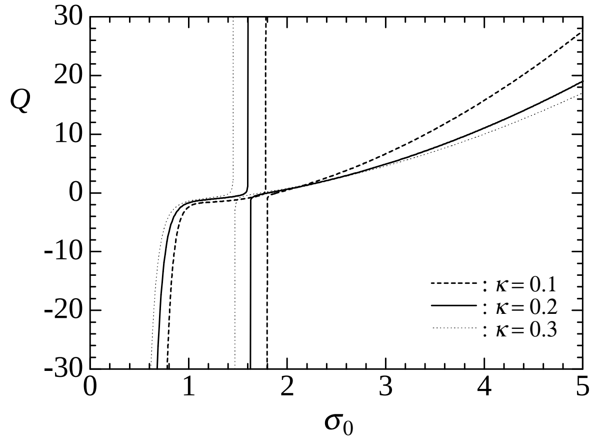

Figure 3: Graphs of Q versus σ_0 with κ fixed at 0.1, 0.2 and 0.3.

$$Q \approx \frac{\pi^3(\gamma + 1 + \kappa)^2}{8\gamma^2\sigma_1(k_2^2 - 4\pi^2)}, \tag{37}$$

where $k_2^2 = (2\pi\sigma_1)^2(1+\kappa s_2)$ and $s_2 = \sigma_0^2/(\sigma_0^2 - 4\sigma_1^2)$ ($\sigma_1 \neq \sigma_0/2$).

The amplitude equation is put in a form convenient to compare with the experiments by using the complex amplitude P of the excess pressure in the tube as

$$\frac{\rho_0 a_0 u_0 p'}{p_0} = \varepsilon\gamma p' = \frac{1}{2}\cos[\pi(x-1)]P\mathrm{e}^{\mathrm{i}\pi\sigma_1 t} + c.c. + ..., \tag{38}$$

where $P = 2\varepsilon\gamma\alpha$ ($\ll 1$) to the lowest order. Then (33) is reduced, on noting $\partial P/\partial t = \varepsilon^2\partial P/\partial t_2$, to

$$\mathrm{i}\mu\frac{\partial P}{\partial t} + SP + \mathrm{i}D|P|P + Q|P|^2 P = \Gamma\mathrm{e}^{\mathrm{i}\pi\Delta\sigma t}, \tag{39}$$

where $D = 64\kappa\delta_J\sigma_1 s_1^3/9\pi\gamma\Omega$ and $\Gamma = 2\pi\gamma\sigma_1 c$.

Frequency response

The frequency response of the stationary oscillations is obtained by seeking a steady solution to (39) on setting $P = |P|\exp(\mathrm{i}\pi\Delta\sigma t)$, $|P|$ being constant. Then it is given by

$$\Delta\sigma = \frac{1}{\pi\mu}\left[S_{re} + Q|P|^2 \pm\sqrt{\frac{\Gamma^2}{|P|^2} - (S_{im} + D|P|)^2}\right]. \tag{40}$$

Table 1: Geometry of the tubes

Geometry	unit	Tube I	Tube II
Tube length: l	[mm]	3256	922.5
Tube diameter: $2R$	[mm]	80.0	40.4
Cavity volume: V	[cm^3]	49.7	30.8
Throat length: L	[mm]	35.6	21.5
Throat diameter: $2r$	[mm]	7.11	8.50
Axial spacing: d	[mm]	50.0	145.0
Size parameter: κ	[−]	0.198	0.166

Table 2: Experimental conditions

Conditions	unit	Tube I	Tube II
Temperature of gas in equilibrium: T_0	[°C]	25.6	26.3
Pressure of gas in equilibrium: p_0	[MPa]	0.1007	0.1017
Sound speed: a_0	[m/s]	347.1	347.5
Natural frequency of resonator: $\omega_0/2\pi$	[Hz]	242.6	444.8
Kinetic viscosity of gas: ν	[m^2/s]	1.567 $\times 10^{-5}$	1.558 $\times 10^{-5}$
Prandtl number: Pr	[−]	0.7089	0.7086

(The ratio of specific heats γ is taken as constant 1.402.)

This suggests that $\Delta\sigma$ is shifted downward by S_{re} (< 0) for the linear dispersion due to the wall friction while it is shifted by nonlinearity upward or downward depending on the sign of Q. For $\Omega \gg 1$, Q is positive so that the nonlinearity gives rise to the upshift. Incidentally the linear response corresponds to the case where $|P| \ll 1$ so that $Q|P|^2$ and $D|P|$ are dropped.

The peak amplitude is determined as

$$P_{\rm peak} = \frac{1}{2D}(-S_{im} + \sqrt{S_{im}^2 + 4D\Gamma}). \qquad (41)$$

In (40), the jet loss $D|P|$ enhances the linear loss S_{im}. For $\Gamma \ll S_{im}^2/4D$, $P_{\rm peak}$ is given by Γ/S_{im}, whereas for $\Gamma \gg S_{im}^2/4D$, it is given by $\sqrt{\Gamma/D}$. The latter suggests that the peak amplitude does not increase in proportion to Γ but to $\sqrt{\Gamma}$. Thus reduction of D is of vital importance to achieve higher peak amplitude.

The frequency response thus derived is now checked against the experiments. Geometry of the tube and experimental conditions are tabulated in Tables 1 and 2 in the column designated as Tube I. The gas column is excited by oscillating bellows mounted at one end of the tube where the greatest and smallest inner diameters of the bellows are 110 mm and 80 mm, respectively. The displacement amplitude of the bellows X_b is fixed at 0.5, 1.5, 2.5 and 3.5 mm, respectively.

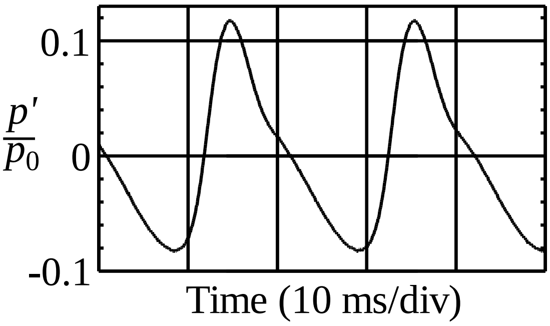

Figure 4: Shock-free pressure profile measured at the closed end of Tube I.

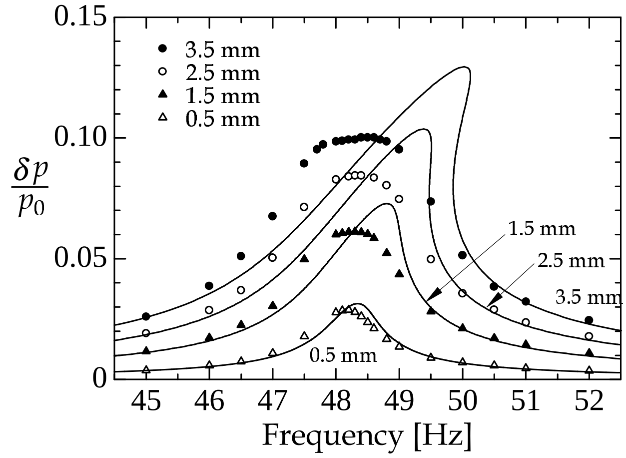

Figure 5: Frequency response of the gas column in Tube I.

At first, we show the shock-free oscillations are achieved at the peak pressure about 10% of the ambient (atmospheric) pressure. Figure 4 shows the temporal profile of the excess pressure on the flat plate at the largest excitation $X_b = 3.5$ mm. The profile is obviously shock-free and different from a sinusoidal form. It is offset upward because of the emergence of the steady pressure distribution.

Figure 5 shows the frequency response where the half of the peak-to-peak value of the excess pressure measured on the flat plate, δp, relative to p_0 is drawn versus the frequency of excitation $\omega/2\pi$. The blank triangle, solid triangle, blank circle and solid circle indicate the data measured at the displacement amplitude of the bellows $X_b = 0.5$, 1.5, 2.5 and 3.5 mm, respectively.

The solid curves show the corresponding response derived by (40). The dimensional frequency $(\sigma_1 + \Delta\sigma)a_0/2l$ is taken as the abscissa while $|P|/p_0$ is taken as the ordinate. For $\kappa = 0.198$, we have $\sigma_0 = 4.55$ and $\sigma_1 = 0.911$. The coefficients in the amplitude equation take the following values: $\mu = 2.43$, $-S_{re} = S_{im} = 0.0430$, $D = 0.398$ and

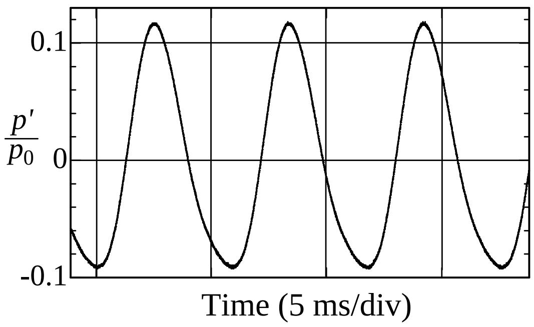

Figure 6: Temporal pressure profile measured at the closed end of Tube II.

$Q = 15.2$. The amplitude X_b is converted to the piston amplitude by $X_p = 1.442X_b$ [8]. This correspondence is derived by equating volume displaced by the bellows with the one by the piston. As X_b is increased from 0.5 mm, the discrepancy becomes pronounced between the theory and the experiments.

Higher harmonic resonance and evanescence

We check conditions for the theory to be valid. The condition whether the tube is consonant or dissonant is concerned with the relation among the frequencies. When the oscillations are regarded as a result of interactions between right- and left-going waves, the wavenumber k is required to take $m\pi$ $(m = 0, 1, 2, ...)$ by the boundary conditions. Thus the tube is still consonant with respect to the wavenumber with $m = 1$ and therefore the condition of dissonance must be considered additionally.

When the lowest mode with $\sigma = \sigma_1$ and $k = \pi$ is excited, there appear the zero harmonics (steady distribution with $\sigma = 0$) having wavenumber $k = 2\pi$ and $k = 0$. Furthermore there appears the third-harmonics having $\sigma = 3\sigma_1$, and $k = 3\pi$ and $k = \pi$. In a special case, such harmonics with frequency $n\sigma_1$ and wavenumber $m\pi$ $(0 \le m < n)$ satisfy the relation (27):

$$(m\pi)^2 = (n\pi\sigma_1)^2 \left[1 + \frac{\kappa\sigma_0^2}{\sigma_0^2 - (n\sigma_1)^2}\right], \qquad (42)$$

where the case with $m = n \neq 0$ is not satisfied. If σ_1 satisfies (42), higher-harmonic resonance occurs so that the energy pumped into the lowest mode is transferred to the higher mode and no increase in amplitude of the fundamental mode is expected. Then the expansion (30) breaks down mathematically.

The second harmonics with $n = 2$ satisfy (42) with $m = 0$, i.e. $2\sigma_1 = \sqrt{1+\kappa}\sigma_0$, if $\sigma_0 = \sqrt{4(3-\kappa)/3(1+\kappa)^2}$. Then the first-order solution and therefore Q diverge. The third harmonic resonance with $n = 3$ and $m = 1$ occurs if $\sigma_1 = \sqrt{\sigma_0/3}$,

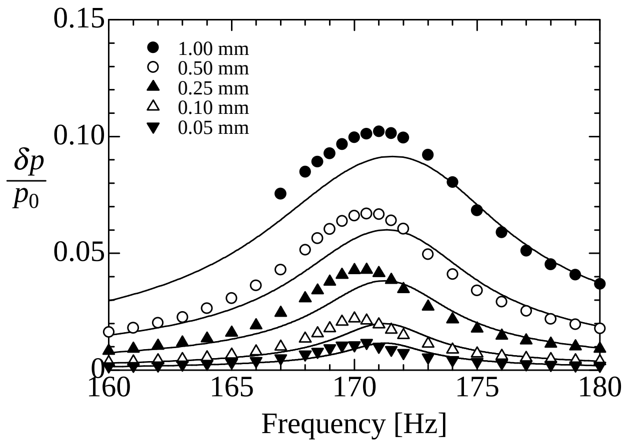

Figure 7: Frequency response of the gas column in Tube II.

i.e. $\sigma_0 = (5 \pm \sqrt{16 - 9\kappa})/3(1+\kappa)$. In this case, although Q is not affected directly, this resonance should be avoided admittedly in order to make the expansion (27) uniformly valid up to any order.

In addition to the higher-harmonic resonances, higher-harmonic 'evanescence' occurs if $n\sigma_1$ hits σ_0. For $n = 2$, this occurs when $\sigma_0 = \sqrt{12/(3+4\kappa)}$, and for $n = 3$, when $\sigma_0 = \sqrt{72/(8+9\kappa)}$. When the frequency of the higher harmonics hits σ_0, Q remains finite but energy would be dissipated in the resonators. This is obviously against increase in amplitude.

In view of these conditions, we look at the experiments. Although $2\sigma_1$ is not equal to σ_2^- in rigorous sense, the ratio σ_2^-/σ_1 in the experiments is 1.98 and very close to 2. Whereas emergence of the shock is suppressed, this is considered to be the main cause of the discrepancy between the theory and the experiments. Even worse, the fifth harmonics $5\sigma_1$ hits σ_0.

Taking account of the above restrictions, another experiment is made by using Tube II in the Table 1 under the condition in Table 2. The bellows are driven at the amplitude $X_b = 0.05$, 0.10, 0.25, 0.50 and 1.0 mm where the greatest and smallest diameters of the bellows are 60 mm and 40 mm, respectively, and X_p is set equal to $1.583X_b$. Figures 6 shows the pressure profile measured on the flat plate at $X_b = 1.0$ mm. The profile is smooth and close to a sinusoidal waveform.

Figure 7 compares the frequency responses by the theory and experiments where the curves for the theory measure $|P|/p_0$ as the ordinate and $\sigma_0 = 2.36$, $\sigma_1 = 0.915$, $\mu = 2.46$, $-S_{re} = S_{im} = 0.0466$, $D = 1.14$, and $Q = 1.99$. The measrued data almost coincide with the theory up to the peak pressure 10%. But the peak amplitudes are underestimated whereas the peak frequencies are overestimated as in Fig. 5. The discrepancy may be attributed firstly to difference between δp and $|P|$, secondly to correspondence between X_b and X_p, and thirdly to semiempirical evaluation of the jet loss. But the higher harmonic components make little contribution to increase in the peak amplitude. Rather the second and third points are questionable. In any case, it is promising to observe that the experimental results tend to agree with the theory up to such a level of excitation.

Discussions

In this section, some discussions are given on difference between the present method and the one due to Lawrenson *al.* [6] and Ilinskii *et al.* [7]. It is common to both methods that the tube becomes dissonant in consequence. But physical mechanisms underlying are different. In the tube with the array, the phase velocity becomes frequency-dependent whereas in the tube of non-uniform cross-section, the harmonic wave both in time and space cannot be propagated and the nonlinear propagation is characterized by characteristics defined by $\mathrm{d}x/\mathrm{d}t = u \pm a$. In this respect, the acoustic system is hyperbolic.

When the tube is shaken, a is given in dimensional form as

$$a^2 = a_0^2 - (\gamma - 1)\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g(t)x\right], \tag{43}$$

$g(t)$ being the acceleration due to the shaker [7], and becomes dependent of not only on the position and but also of the time explicitly rather than the frequency. Nevertheless the system still remains hyperbolic with the characteristics given by $\mathrm{d}x/\mathrm{d}t = u \pm a$. Therefore there remains a possibility of emergence of a shock.

In the present method, in contrast, there exist no characteristics in (25) and the acoustic system is dispersive not hyperbolic. Therefore no discontinuity is now allowed even in the linear theory. But as ε becomes large, competition between ε and κ comes into play and the ratio κ/ε becomes an important parameter. In the experiments described, this ratio takes values of about unity. But it is now open as to what is the smallest value of this ratio for shock-free oscillations to be achieved when κ is fixed.

Conclusion

For resonant excitation of a gas column confined in a tube, this paper has demonstrated that connection of the array of Helmholtz resonators is useful to achieve shock-free, high-amplitude oscillations up to the peak pressure 10% of the ambient pressure. The cubic nonlinear theory has been developed to derive the frequency response and this has been checked against the experiments. When the higher harmonic resonance and evanescence are avoided, it has been revealed that the experimental results tend to agree with the theory quantitatively.

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