SPECTRUM OF ACOUSTIC CAVITATION

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Abstract

Individual spectral bands of the cavitation radiation are characterized by a finite width and even by a definite shape. So far, the nature of these measurable characteristics has been poorly understood. A step toward the generalization of the conventional models was the inclusion of noise, i.e., the analysis of the nonlinear dynamics of a bubble in the field of an intense harmonic signal in the presence of a random component. The effect of fluctuations associated with the random field component is found to be most pronounced in the vicinity of the bifurcation values of the field amplitude and detuning, these values corresponding to changes in the number of stable oscillatory states of a bubble. Behavior of the bubble in this region is characterized by a significant increase in the duration of transition processes and by an essentially non-Gaussian distribution of fluctuations in the vicinity of stable trajectories.

Introduction

The spectrum of the radiation caused by acoustic cavitation in liquid has the form of single bands rising above a noise base [1]. Individual spectral bands of the cavitation radiation are characterized by a finite width and even by a definite shape (see Figure 1). The positions of the bands correspond to harmonics, subharmonics, and ultrasubharmonics of the excitation frequency. The presence of single bands in the spectrum is related to the strongly nonlinear dynamics of single gas bubbles that occur in the field of an intense acoustic wave. The commonly accepted explanation for the presence of the noise base is the generation of short pulses accompanying the collapse of single inclusions. The specific features of the band shown in Figure 1 (for definiteness we consider the spectrum in the vicinity of the fundamental frequency), namely, the narrow components rising over a wide base, the asymmetric base deviating from the Lorentz form retain its shape in widely different experimental conditions [1-2].

The real spectrum of the acoustic pressure that causes oscillation of single bubbles in the sheet noticeably differs from the spectrum used in theoretical calculations, which usually takes into account only the fundamental harmonic. A step toward the generalization of the conventional model was the inclusion of noise, i.e., the analysis of the nonlinear dynamics of a bubble in the field of an intense harmonic signal in the presence of a random component of much lower intensity [3].

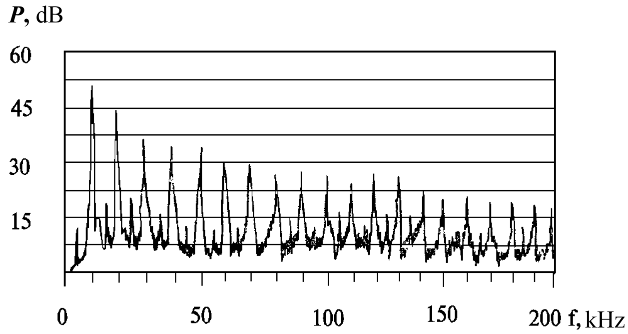
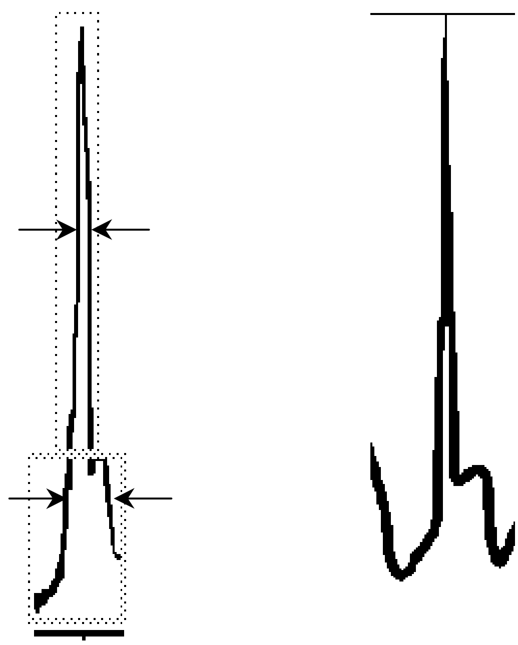


Figure 1. Spectrum of acoustic cavitation [1]

Asymmetry



Akulichev, (1968); $f = 10$ kHz, water   Lauterborn & Holzfuss (1986), $f = 22.6$ kHz, water

Definitions and equations

In the Rayleigh–Plesset equation describing radial pulsation of a gas bubble, the presence of a noise component is taken into account by an additional term entering into the expression for an external field acting upon the bubble:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{P_0}{\rho_0}\left[1 - \left(\frac{R_0}{R}\right)^{3\gamma}\right] + 2\delta R_0 \dot{R} = \frac{P_0 - P(t)}{\rho_0}, \quad (1)$$

$$P(t) = P_0 + p_m \sin(\omega_p t) + p_N(t).$$

Here, P_0, ρ_0, P, R_0, R represent the equilibrium and current values of the pressure, the density of the liquid, and the bubble radius; p_m, and ω_p are the amplitude and frequency of the high-power harmonic signal component, respectively, and $p_N(t)$ is the random noise, γ is the polytropic index; and δ is the damping constant. In what follows, we will use the simplest model, assuming that $p_N(t)$ can be described as a Gaussian delta-correlated random process. Solving Eq. (1) with the use of an asymptotic expansion in the small parameter $|R-R_0|/R_0<<1$ we arrive at a system of "reduced" equations for slowly varying amplitudes a and phases ϑ.

$$\frac{(R-R_0)}{R_0}=\frac{1}{2}\left(ae^{-i(\omega_p t+\vartheta)}+c.c.\right)+\varepsilon u_1(a,\vartheta,t)+\varepsilon^2 u_2(a,\vartheta,t)+...$$

Here, ε is a small dimensionless parameter introduced for describing the order of the nonlinear terms and $u_1(a,\vartheta,t)$ and $u_2(a,\vartheta,t)$ are the higher-order terms of the expansion. Correct to third-order terms, we can perform the analysis in the vicinities of the fundamental resonance, the first and second harmonics, and the first and second subharmonics. Generally speaking, the structure of the "reduced" equations in each of these regions should be different. Expressing this system in terms of the variables $u(t)=a(t)\cos[\vartheta(t)]$, $v(t)=a(t)\sin[\vartheta(t)]$ and $p_N(t)=\cos(\Omega_0 t)\cdot\pi_N(t)+\sin(\Omega_0 t)\cdot\overline{\pi}_N(t)$, we obtain in the vicinity of the fundamental resonance $\omega_p\sim\Omega_0$ ($\Omega_0=(3\gamma P_0/\rho_0 R_0^2)^{1/2}$ is the fundamental frequency).

$$\frac{du}{dt}=\frac{p_m+\overline{\pi}_N(t)}{2\rho_0\omega_p R_0^2}-\delta u+(\omega_p-\Omega_0)v+\kappa\Omega_0 v(u^2+v^2),$$

$$\frac{dv}{dt}=-\delta v-(\omega_p-\Omega_0)u-\kappa\Omega_0 u(u^2+v^2)+\frac{\pi_N(t)}{2\rho_0\omega_p R_0^2}. \quad (2)$$

The system of Eqs. (2) is an example of stochastic differential equations. The description of the evolution of this system is based on the solution of the Einstein–Fokker–Planck (EFP) equation for the probability density of the dynamic states $W(u,v)=<\delta(u-u(t))\delta(v-v(t))>$, where the averaging is performed over the random force ensemble [3]. For the case under study, the EFP equation has the form

$$\frac{\partial W}{\partial t}+\frac{\partial}{\partial u}\left(PW\right)+\frac{\partial}{\partial v}\left(QW\right)=D\left(\frac{\partial^2 W}{\partial u^2}+\frac{\partial^2 W}{\partial v^2}\right);$$

$$D=\frac{\pi}{4}\Omega_0\frac{S(\Omega_0)\Omega_0}{\left(\rho_0 R_0^2\Omega_0^2\right)^2}, \quad (3)$$

$$<\pi_N(t)\pi_N(t+\tau)>=<\overline{\pi}_N(t)\overline{\pi}_N(t+\tau)>=2\pi S(\Omega_0)\delta(\tau),$$

$$P(u,v)=\frac{p_m}{2\rho_0\omega_p R_0^2}-\delta u+(\omega_p-\Omega_0)v+\kappa\Omega_0 v(u^2+v^2),$$

$$Q(u,v)=-\delta\cdot v-(\omega_p-\Omega_0)u-\kappa\Omega_0 u(u^2+v^2).$$

In fact, it is easy to obtain a stationary solution to Eq. (3) in the region of the existence of a single solution to Eq. (2) in the absence of random force ($P(u_\bullet,v_\bullet)=0$, $Q(u_\bullet,v_\bullet)=0$) and this solution has a Gaussian form

$$W_{eq}=\exp\left(-\frac{\Phi(u,v)}{D}\right), \quad (4)$$

$$\Phi(u,v)=\Phi_0+\left[\frac{(u-u_*)^2}{2c_{11}}+\frac{(u-u_*)(v-v_*)}{c_{12}}+\frac{(v-v_*)^2}{2c_{22}}\right],$$

$$c_{11}^{-1}=\Lambda(b_{12}^2+b_{22}^2+b_{11}b_{22}-b_{12}b_{21}),$$

$$c_{22}^{-1}=\Lambda(b_{11}^2+b_{21}^2+b_{11}b_{22}-b_{12}b_{21}),\ c_{12}^{-1}=\Lambda(b_{11}b_{12}+b_{21}b_{22}),$$

$$\Lambda=(b_{11}+b_{22})/[(b_{11}+b_{22})^2+(b_{12}-b_{21})^2];$$

$$b_{11}=\left(\partial P(u,v)/\partial u\right)_{u=u_*,v=v_*},\ b_{12}=\left(\partial P(u,v)/\partial v\right)_{u=u_*,v=v_*}$$

$$b_{21}=\left(\partial Q(u,v)/\partial u\right)_{u=u_*,v=v_*}\quad b_{22}=\left(\partial Q(u,v)/\partial v\right)_{u=u_*,v=v_*};$$

here, $\Phi_0=D\ln[2\pi D/(c_{11}^{-1}c_{22}^{-1}-c_{12}^{-2})]$ is the normalization constant determined from the condition $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dudvW_{eq}(u,v)=1$. Using the probability density W_{eq} in the form obtained above, we determine the mean and the mean square values of the dynamic variables

$<u>=u_*$, $<v>=v_*$, $<a>=a_*$, $<\vartheta>=\vartheta_*$

$$<(a-<a>)^2>=\frac{D}{\delta}\times$$
$$\times\frac{\delta^2+\left[\omega_p-\Omega_0+\kappa\Omega_0 a_*^2\right]\left[\omega_p-\Omega_0+2\kappa\Omega_0 a_*^2\right]}{\delta^2+\left[\omega_p-\Omega_0+\kappa\Omega_0 a_*^2\right]\left[\omega_p-\Omega_0+3\kappa\Omega_0 a_*^2\right]},$$

$$a_*^2<(\vartheta-<\vartheta>)^2>=\frac{D}{\delta}\times$$
$$\times\frac{\delta^2+\left[\omega_p-\Omega_0+2\kappa\Omega_0 a_*^2\right]\left[\omega_p-\Omega_0+3\kappa\Omega_0 a_*^2\right]}{\delta^2+\left[\omega_p-\Omega_0+\kappa\Omega_0 a_*^2\right]\left[\omega_p-\Omega_0+3\kappa\Omega_0 a_*^2\right]},$$

$$a_*<(a-<a>)(\vartheta-<\vartheta>)>=-\frac{D}{\delta}\times$$
$$\times\frac{\delta\kappa\Omega_0 a_*^2}{\delta^2+\left[\omega_p-\Omega_0+\kappa\Omega_0 a_*^2\right]\left[\omega_p-\Omega_0+3\kappa\Omega_0 a_*^2\right]};$$

$$a_*^2\left[\left(\omega_p-\Omega_0+\kappa\Omega_0 a_*^2\right)^2+\delta^2\right]=\frac{p_m^2}{4\rho_0^2\omega_p^2 R_0^4},$$

$$\cos\vartheta_*=\frac{2\rho_0 R_0^2\omega_p^2(\delta/\Omega_0)}{p_m}a_*. \quad (5)$$

In addition, special consideration should be given to the bistability region of nonlinear bubble pulsation, in particular, to the vicinities of the bifurcation curves of the dynamic system (2), i.e., the curves on which the denominator in Eqs. (5) becomes zero. This problem has been investigated in [2].

Spectral density

The shape of the radiation spectrum is exclusively determined by the contribution of the autocorrelation function of the radiation. Substituting the asymptotic expansion for the bubble radius and retaining only the leading terms, we separate naturally the coherent and incoherent contributions to the spectral density.

$$S_r(\mathbf{r},\omega)=\frac{\rho_0^2 R_0^6\omega_p^4}{2\pi\,|\mathbf{r}-\mathbf{r}'|^2}\int\limits_{-\infty}^{+\infty} d\tau\, e^{-i\omega\tau} < a(t)\cos(\omega_p t+\vartheta(t))\times$$
$$\times a(t+\tau)\cos[\omega_p(t+\tau)+\vartheta(t+\tau)]>=$$
$$=\frac{R_0^2}{|\mathbf{r}-\mathbf{r}'|^2}\left(\rho_0^2 R_0^4\omega_p^4\right)\left[\tilde{S}_c(\omega)+\tilde{S}_n(\omega)\right] \qquad (6)$$

$$\tilde{S}_c(\omega)=\frac{1}{4}\left(<u>^2+<v>^2\right)\left[\delta(\omega+\omega_p)+\delta(\omega-\omega_p)\right]=$$
$$=\frac{a_*^2}{4}\left[\delta(\omega+\omega_p)+\delta(\omega-\omega_p)\right],$$

$$\tilde{S}_n(\omega)=\frac{1}{4}\Big\{\tilde{S}_{uu}(\omega-\omega_p)+\tilde{S}_{uu}(\omega+\omega_p)+\tilde{S}_{vv}(\omega-\omega_p)+$$
$$+\tilde{S}_{vv}(\omega+\omega_p)+i\left[\tilde{S}_{vu}(\omega-\omega_p)-\tilde{S}_{uv}(\omega-\omega_p)\right]+$$
$$+i\left[\tilde{S}_{vu}(\omega+\omega_p)+\tilde{S}_{uv}(\omega+\omega_p)\right]\Big\}.$$

Here, $\tilde{S}_{uu}(\omega)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} d\tau\, e^{-i\omega\tau} <\tilde{u}(t)\tilde{u}(t+\tau)>$ and $\tilde{u}=u-<u>$. The correlators of other types are calculated in a similar way.

Outside the vicinity of the singularities, (mentioned above), where the amplitude and phase fluctuations can be described by the Ornstein–Zernicke process [4], the expression for the incoherent contribution to the spectral density can be reduced to the following form (for definiteness, we select the vicinity $\omega \sim \omega_p$):

$$\tilde{S}_n(\omega)=\frac{D}{2}\times \qquad (7)$$
$$\frac{\left[(\omega-\omega_p)^2+(\omega-\omega_p)(b_{12}-b_{21})+\frac{1}{2}(b_{11}^2+b_{12}^2+b_{21}^2+b_{22}^2)\right]}{\left[(\omega-\omega_p)^2-(b_{11}b_{22}-b_{12}b_{21})+i(\omega-\omega_p)(b_{11}+b_{22})\right]\times[c.c.]},$$

where (see definition (4)) $b_{11}=-\delta+2\kappa\Omega_0 u_* v_*$, $b_{12}=(\omega_p-\Omega_0)+\kappa\Omega_0(u_*^2+3v_*^2)$, $b_{22}=-\delta-2\kappa\Omega_0 u_* v_*$, $b_{21}=-(\omega_p-\Omega_0)-\kappa\Omega_0(3u_*^2+v_*^2)$,. In the strongly nonlinear regime under study, when $\kappa\Omega_0 a_*^2 \geq \delta$, this shape widely deviates from the Lorentz one $\tilde{S}_n(\omega)\approx (D/2)\left[(\omega-2\omega_p+\Omega_0)^2+\delta^2\right]^{-1}$, to which it is reduced when the nonlinear terms are neglected. An important fact is that the magnitude and the halfwidth of the spectral density of radiation are determined by the stability of the nonlinear dynamic state of the bubble, because $(b_{11}b_{22}-b_{12}b_{21})$ is the product of the corresponding linear stability exponents of system (2) in the absence of noise. The effect of fluctuations related to the random component of the acoustic field is most significant in the vicinity of the bubble wall bifurcations, which correspond to a change in the number of stable oscillatory states of the bubble.

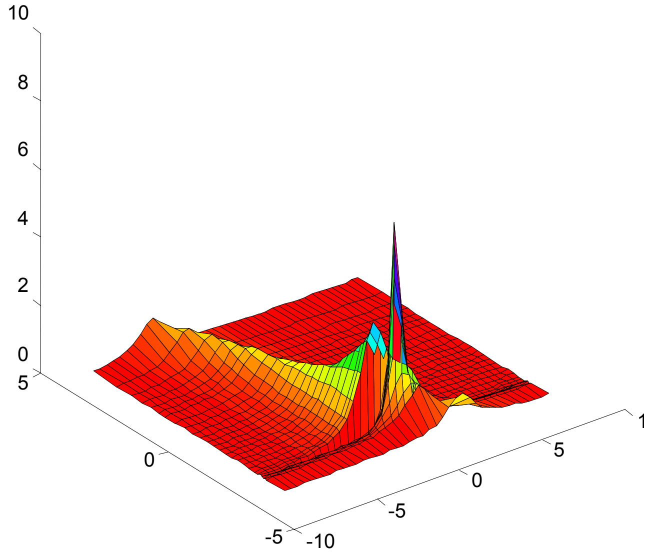


Figure 2. Spectral density of the bubble radiation S_N ($S_N=(\delta^2/D)\tilde{S}_n$) as a function of the dimensionless variables $\eta=(\omega_p-\Omega_0)/\delta$ and $\Delta=(\omega-\omega_p)/\delta$ at $p_m=1.4p_k$. The values of η are plotted along the *y*-axis, and Δ is represented by the *x*-axis.

Numerical method

To overcome the deficiency of analytical solution based on the asymptotic expansion, we have studied the regime corresponding to bistable oscillatory states of the bubble by numerical techniques [4]. The results of solving Eq. (1) are presented in the form of the radius-time dependence $x(t)\equiv\left((R(t)-R_0)/R_0\right)$ and a map of the phase portraits. Conventional approach in analyzing bifurcations in the nonlinear oscillations of a bubble is to study dependence of the maximum radius of a bubble on the control parameters. This characteristic is not a Poincaré map during the initial period of time, when the evolution is determined by the transition processes, but it becomes such a map upon attaining the steady-state regime. Convenience of this variable is related to the fact that, considered as a function of detuning, it describes the amplitude-frequency characteristic of the bubble pulsation in the approximation of weak nonlinearity, which makes possible direct comparison of the numerical data to the results obtained by approximate analytical methods.

Figure 3 shows the results of calculation of the distribution density of the maximum radius $f(x_{\max},\eta)$ for the values of detuning $\eta=\Omega_0^2(R_0)/\omega_p^2-1$ from the interval (–0.26, 0.2) in the region of the

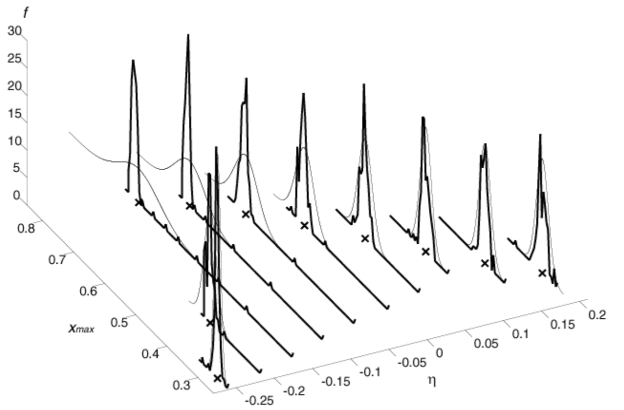


Figure 3. The distribution density of maximum radius of bubble pulsation

fundamental resonance, quality factor $Q = \Omega_0 / \delta$ =10; external field amplitude $(p_m / p_0) \equiv s = 1{,}1$; and noise intensity $(\overline{p_N^2} / P_0^2)^{1/2} = 0.1$. The distribution density is defined as $f(x_{\max}, \eta) \equiv \left[N(x_{\max}) / (N \Delta x) \right]$, where $N(x_{\max})$ is the number of values of the maximum radius in the interval $[x_{\max} - \Delta x/2, x_{\max} + \Delta x/2]$, $x_{\max} \equiv (R_{\max} - R_0)/R_0$, and N is the total number of $R_{\max}$ values in the time interval under consideration (in this case, 400 periods). For the reference, thin lines depict the Gaussian distributions characterized by the same mean values and dispersions as the $x_{\max}$ series for which the distribution function is constructed. In the (x, η) plane, the markers indicate the values of $x_{\max}$ (maximum radius) for the steady-state oscillations in the function absence of a random force, which can be compared to the known data.

One should note, that obtained results describe bubble dynamics for a fixed realization of the random variable p_N. In order to find stable, mean characteristics one should average the distribution density over an ensemble of realization of the random variable. To avoid this cumbersome procedure we consider self-averaged quantities, in particular, K – an entropy

$$K = -\sum_{i=0}^{N} \Delta x f(x_{\max}^i) \ln[f(x_{\max}^i) \Delta x], \qquad (8)$$

here the summation is carried out over all values of the maximum radius in the series produced by numerical solution of Eq. (1).

Conclusions

We can suggest the following physical interpretation of the results presented above. Outside the region of bistability, the effect of the noise component on the character of pulsation is small, the duration of transition processes is short, and the fluctuations quite rapidly begin to follow the Gaussian distribution, in agreement with the results of the analytical description. In the case of two stable states, the character of pulsation of the bubble—at least, within a not too long interval of time (400 periods)—significantly differs from the physical pattern adopted in [3], when the system sufficiently rapidly reaches one of the two equilibrium states, performs small fluctuations at this equilibrium position, and exhibits rare transitions between the two equilibrium states. As can be seen from Figure 3, the distribution function has a complicated profile and significantly differs from Gaussian, which is evidence of a low stability of pulsation and a significant increase in the duration of transition processes.

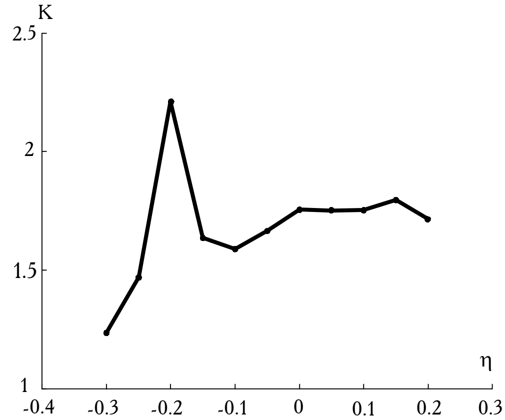


Figure 4. The entropy as function of the detuning $\eta = \Omega_0^2(R_0)/\omega_p^2 - 1$ for the bistability region of bubble oscillations. The dots indicate the numerical result at the same values for the parameters as used for calculation of the distribution density (Figure 3).

Acknowledgments

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