NONLINEAR ACOUSTIC WAVES IN A 2D GRANULAR MEDIUM

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Abstract. A two-dimensional (2D) model of a granular medium is elaborated that represents a square lattice consisting of elastically interacting round particles with two translational and one rotational degrees of freedom. The differential equations describing propagation and interaction of waves of various types in such a medium are derived. The structure of the obtained equations is shown to be invariant to the shape and sizes of granules. Variations of the last ones influence only on values of the coefficients. The account of microturns of the particles leads to appearance of a wave of microrotations (spin wave). When microturns are absent, in the linear approximation the governing equations degenerate into Lame equations for anysotropic medium with the cubic symmetry. The analytical relationships between elasticity constants, on the one hand, and the geometrical sizes of particles and parameters of interparticle interactions, on the other hand, have been found.

Introduction. Mechanical properties of granular media (for instance, rock, powder, ceramic material) depend on location and sizes of microparticles, as well as on forces of interaction between them. The main goal of a mathematical modelling is derivation of continuous equations of motion and equations of state that are able to describe adequately a discrete nature of the medium. A number of researches on dynamical behaviour of granular media with regular and random structure without allowance for particle rotation were undertaken in [1-3]. Models for granular media with account of particle rotation were considered in [4-5]. However, in spite of a rich variety of theories of complex structure media, new models of dynamical behaviour of granular media are still needed. Such models should satisfy the following conditions:

- to contain a minimal amount of new parameters leading to qualitatively new results;
- to enable to find relationships between micromodel parameters and physical-mechanical characteristics of a medium (porosity, elasticity constants and etc.);
- to degenerate, in the limiting cases, into classical theories of solids.

Below consider a two-dimensional (2D) model of a granular medium that satisfies the above conditions.

1. The discrete model

A granular material can be regarded as a continuous medium consisting of two components (phases). One component is rigid particles having mass M and inertia J, the other one is a porous space between the particles (fig. 1). In the model at issue, the porous space is a zero-mass nonlinearly elastic medium through which force and moment interactions are translated.

A 2D square lattice consisting of homogeneous round particles (granules) having mass M and diameter $2d$. At the initial state (t=0), they are located in the lattice nodes, and the distance between the mass centres of neighbouring granules both along the x- and y-axes equals a (fig. 1). Each particle, when moves in its plane, has three degrees of freedom: displacement of the mass centre of the particle with number N=(i, j) along the x- and y-axes (translational degrees of freedom $u_{i,j}$ and $w_{i,j}$) and turns relative to the mass centre (rotational degree of freedom $\varphi_{i,j}$) (fig. 2). The kinetic energy of the lattice is equal to the sum of separate particles:

$$T = \sum_i \sum_j \left[\frac{M}{2}\left(\dot{u}_{i,j}^2 + \dot{w}_{i,j}^2 \right) + \frac{J}{2}\dot{\varphi}_{i,j}^2 \right]. \qquad (1)$$

Here $J = Md^2/2$ is its moment of inertia about the axis through the mass centre. The dots denote derivatives with respect to time.

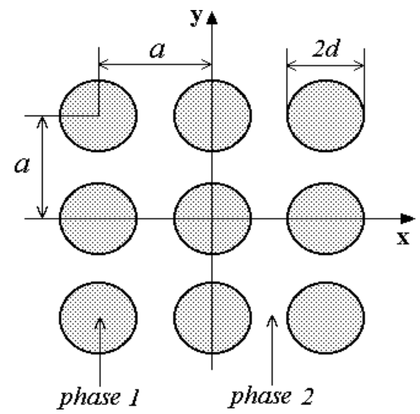

Fig. 1

The particle N is supposed to interact directly with eight nearest neighbours in the lattice. The mass centres of four of them are on horizontal and vertical lines, while the mass centres of the other four neighbouring particles lie on diagonals (fig.2). The first four particles, mass centres of which are located on circumference of radius a, will be called below particles of the first coordination sphere. The rest four neighbours will be called particles of the second coordination sphere (their mass centres lie on circumference of radius $a\sqrt{2}$). The centres of both circumferences coincide with particle N. Interactions are modelled using elastic springs connecting neighbouring particles [5-9]. There are springs of four types. The central springs posses rigidity k_0, while rigidity of the horizontal and vertical springs equals k_1. Together they define force interactions in the material at extension/compression, while the springs with k_1 translate moments at particle rotation. The diagonal springs with rigidity k_2 characterise force interactions of the particles at shear deformations in the material. And the springs modelling interactions with the particles of the second coordination sphere have rigidity k_3. In the figure 2 the springs are numerated according to the following rule: the first digit of a number corresponds to the rigidity parameter of the spring (0, 1, 2, or 3), while the second digit determines its order number among the springs of this type.

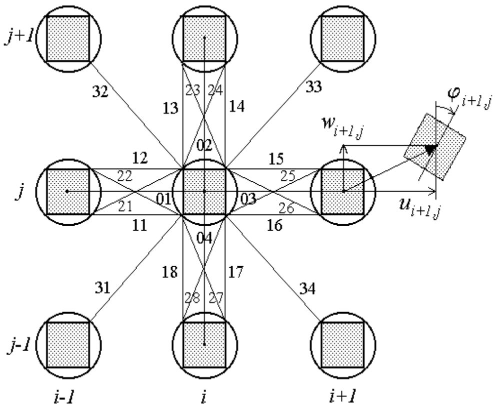

Fig. 2

The particle displacements are assumed to be small with respect to the lattice period, a, and the interaction between the particles depends on relative extensions of the springs that arise when the granules deviate from the equilibrium states (fig. 2). The potential energy concerned with deformations of the springs connecting particle N with eight nearest neighbours in the lattice takes on the form

$$U_N = \frac{k_0}{2}\sum_{n=1}^{4}\left(D_{0n}^2 + \gamma D_{0n}^3\right) + \frac{k_1}{2}\sum_{n=1}^{8} D_{1n}^2 + \frac{k_2}{2}\sum_{n=1}^{8} D_{2n}^2 + \frac{k_3}{2}\sum_{n=1}^{4} D_{3n}^2 . \quad (2)$$

Here D_{jn} are extensions of the springs that are determined by relative variations of distances between the correspondent points of granules [5].

The form of the function U_N was chosen under the assumption that the central springs take the major portion of force, therefore the expression for the energy of the central springs takes into account the cubic term γD_{0n}^3. The springs of three other types with rigidity k_1, k_2, and k_3 are responsible for the relatively weak moment interactions of the particles associated with rotations, and their energy is represented by quadratic terms only. Spring extensions have been calculated when quantities $\Delta u_i = (u_{i,j} - u_{i-1,j})/a \sim \Delta w_i = (w_{i,j} - w_{i-1,j})/a \sim \varphi_{i,j} \sim \varepsilon \ll 1$ and $\Phi_i = \left(\varphi_{i,j} + \varphi_{i-1,j}\right)/2 = \varphi_{i,j} - 0{,}5a\Delta\varphi_i \ll \pi/2$ are supposed to be small [6].

The expression for the potential energy per one particle to an accuracy of ε^3 has the form:

$$\begin{aligned}
U_{ij} = {} & B_1\left((\Delta u_i)^2 + (\Delta w_j)^2\right) + B_2\left((\Delta u_j)^2 + (\Delta w_i)^2\right) + \\
& + 0{,}5d^2 B_3\left((\Delta\varphi_i)^2 + (\Delta\varphi_j)^2\right) + \\
& + B_4(\Delta w_i\Phi_i - \Delta u_j\Phi_j) + B_5\varphi_{ij}^2 + \\
& + B_6(\Delta u_i\Delta w_j + \Delta u_j\Delta w_i) + p_1\left((\Delta u_i)^3 + (\Delta w_j)^3\right) + \\
& + p_2\left(\Delta u_i(\Delta w_j)^2 + (\Delta u_j)^2\Delta w_i\right) + \\
& + p_3(\Delta u_i\Phi_i^2 + \Delta w_j\Phi_j^2) + \\
& + p_4(\Delta u_i\Delta w_i\Phi_i - \Delta u_j\Delta w_j\Phi_j) + \\
& + 3p_5\left(\Delta u_i(\Delta u_j)^2 + (\Delta w_i)^2\Delta w_j\right) - \\
& - p_5(\Delta u_i + \Delta w_j)(2\Delta u_j\Delta w_i + \Delta u_i\Delta w_j) + \\
& + 6p_5(\Delta w_i\Delta w_j - \Delta u_i\Delta u_j)\varphi_{ij} .
\end{aligned} \quad (3)$$

Here, the first two terms describe the energy at longitudinal and shear deformations, the third and fourth terms characterize the energy due to noncentral (moment) interactions of the particles, and the fifth one stands for the energy of coupling between the transverse and rotational degrees of freedom of the particles. All the other terms with coefficients p_n (n=1÷5) describe the energy of nonlinear interactions. Coefficients responsible for linear effects are expressed in terms of micromodel parameters and elasticity constants of the springs by the following:

$$B_1 = a^2\left((1+3\gamma\delta_0)\frac{k_0}{2} + k_1 + \left(\frac{h^2}{r^2} + 2\delta_2\frac{d^2}{r^3}\right)k_2 + \frac{1}{2}\left(1+\frac{\delta_3}{h\sqrt{2}}\right)k_3\right),$$

$$B_2 = a^2\left(\frac{\delta_0}{2h}k_0 + \frac{\delta_1}{h}k_1 + \right.$$

$$+\frac{2d^2}{r^2}k_2+\frac{1}{2}(1+\frac{\delta_3}{h\sqrt{2}})k_3)),$$

$$B_3=a^2\big((1+\frac{\delta_1}{h})k_1+ \qquad (4)$$

$$+(\frac{a^2}{r^2}+\delta_2\frac{(d\sqrt{2}-h)^2}{r^3})k_2+\frac{2\delta_3}{h\sqrt{2}}k_3\big),$$

$$B_4=2ad\sqrt{2}\big(\frac{\delta_1}{h}k_1+$$

$$+(\frac{ad\sqrt{2}}{r^2}-\delta_2\frac{h(d\sqrt{2}-h)}{r^3})k_2+\frac{\delta_3}{h\sqrt{2}}k_3\big),$$

$$B_5=4d^2\big(k_1+$$

$$+(\frac{a^2}{r^2}+\delta_2\frac{(d\sqrt{2}-h)^2}{r^3})k_2+\frac{2\delta_3}{h\sqrt{2}}k_3\big),$$

$$B_6=a^2(1-\frac{\delta_3}{h\sqrt{2}})k_3,$$

where $h=a-d\sqrt{2}$ and $r=\sqrt{2d^2+h^2}$ are lengths of horizontal and diagonal springs, correspondingly, at the initial state. Nonlinearity factors have the form:

$$p_1=a^3\left(\frac{\gamma}{4h}k_0+\frac{2d^2h}{r^4}k_2+\frac{k_3}{4h}\right),$$

$$p_2=\frac{a^3}{h}\big((1+3\delta_0\gamma)\frac{k_0}{4}+\frac{k_1}{2}-$$

$$-\frac{h^2}{r^4}(4d^2-h^2)k_2-\frac{k_3}{4}\big),$$

$$p_3=\frac{2ad^2}{h}\big(\frac{k_1}{2}+$$

$$+\frac{h(d\sqrt{2}-h)}{r^4}(4d^2+3dh\sqrt{2}-h^2)k_2+k_3\big), \qquad (5)$$

$$p_4=\frac{a^2d\sqrt{2}}{h}\left(k_1+\frac{2ah(d\sqrt{2}-h)^2}{r^4}k_2\right),$$

$$p_5=\frac{a^3}{4h}k_3.$$

It is interesting that in the considered approximation the constants of nonlinear interaction, p_1 and p_5, are due basically to the existence of noncentral (moment) particle interactions in a granular medium. Central interactions influence on constant p_1 exclusively by means of the cubic summand in the potential energy of the longitudinal deformation. The size of the microparticles is especially critical for the values of the constants B_4, B_5, p_3, and p_4. So, when the particles are material points (d=0), all these constants are equal to zero. It should be noted that in a one-dimensional chain of rectangular particles [5], if they degenerate into material points and preliminary deformations of the springs are not taken into account, constant B_2 also vanish to zero. Moreover, at certain values of micromodel parameters, constants p_2 and p_3 can be negative, while the other constants (if $\delta_j \ll h$) are always positive.

The differential-difference equations describing nonlinear dynamical processes in the considered system can be obtained from the variational Hamilton's principle by the Lagrange function $L=T-U$. These equations are too tedious to be written in an explicit form, but they are very convenient for numerical simulation of the system response to external dynamical forcing.

3. The continuum model

In typical cases, when the length λ of an acoustic wave is much larger than the period of the lattice ($\lambda \gg a$), one can pass from discrete variables i and j indicating the number of the particle to continuous spatial variables $x=ia$ and $y=ja$, as well as interpolate the functions specified at discrete points by continuous expressions. If only quantities of the first order of infinitesimal are taken into consideration in the expansion, then the density per unit length of the Lagrange function L can be derived from (1) and (3). Using this function a set of differential equations describing the interaction of different types of waves in a granular medium is derived in compliance with the variational Hamilton's principle [6]:

$$u_{tt}-c_1^2u_{xx}-c_2^2u_{yy}-s^2w_{xy}+\beta_1\varphi_y=\frac{1}{2}\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y},$$

$$(6)$$

$$w_{tt}-c_2^2w_{xx}-c_1^2w_{yy}-s^2u_{xy}-\beta_1\varphi_x=\frac{\partial F_3}{\partial x}+\frac{1}{2}\frac{\partial F_4}{\partial y},$$

$$R^2\varphi_{tt}-R^2c_3^2(\varphi_{xx}+\varphi_{yy})+\beta_2\varphi+2\beta_1(w_x-u_y)=F_5.$$

Here F_i $(i=1\div 5)$ are nonlinear parts of these equations [7], $c_i=\sqrt{2B_i/\rho a^2}$ $(i=1\div 3)$ are the velocities of longitudinal, transverse, and rotational waves, $\beta_1=B_4/\rho a^2$ and $\beta_2=2B_5/\rho a^2$ are dispersion parameters, $s=\sqrt{2B_6/\rho a^2}$ is the constant of linear coupling between longitudinal and transverse waves (deformations), $\alpha_j=2p_j/\rho a^2$ $(j=1\div 5)$ are nonlinearity factors, $\rho=M/a^2$ is the surface density of the granular medium, $R=\sqrt{J/M}=d/\sqrt{2}$ is the radius of inertia of medium microparticles relative to the mass centre.

If the rotational degree of freedom is not considered, the linear parts of derived equations are reduced to the Lame equations for cubic-symmetry media. In addition to the known factors in linear (c_1, c_2, and s) and nonlinear (α_1, α_2, and α_5) parts of equations, five constants, which are not known in the classical theory of elasticity, appear in a granular medium (including three linearity constants (c_3, β_1, and β_2) and two nonlinearity parameters (α_3 and α_4). From relations (8) and (6) it is visible, that account of interaction of each particle with the particles of the second coordination sphere is very important. Constants s and α_5 vanish to zero without account of such interactions. It should be noted that if the distance between the particles tends to zero ($h \to 0$) then influence of diagonal springs (i.e. noncentral interactions) on the longitudinal wave velocity c_1 decreases.

3. Relationships between wave characteristics and micromodel parameters

In order to get an insight into the relationship between micro-and macroproperties of a granular medium we need to analyse the dependence of acoustic wave characteristics of a medium on micromodel (microstructure) parameters involving the force constants k_0, k_1, k_2, and k_3 and geometric characteristics: distance between centres of the particles a (the lattice period) and diameter of particles $2d$. As follows from (4) and (5), the coefficients of equations (6) are expressed in terms of microstructure constants by the following way:

$$\rho c_1^2 = k_0 + 2k_1 + \frac{2h^2}{r^2} k_2 + k_3,$$
$$\rho c_2^2 = \frac{2d^2}{r^2} k_2 + k_3, \quad \rho c_3^2 = 2\left(k_1 + \frac{a^2}{r^2} k_2 \right),$$
$$\rho\beta_1 = 8d^2 \left(\frac{k_1}{a^2} + \frac{k_2}{r^2} \right), \quad \rho\beta_2 = \frac{4d^2}{r^2} k_2, \quad (7)$$
$$\rho s^2 = 2k_3.$$

On the other hand, the microstructure parameters can be expressed in terms of acoustic wave characteristics:

$$k_0 = \rho\left(c_1^2 + \frac{a^2 + h^2}{2d^2}\beta_2 - \frac{a^2}{4d^2}\beta_1 + \frac{s^2}{2}\right),$$
$$k_1 = \rho(\beta_1 - 2\beta_2)\frac{a^2}{8d^2}, \quad (8)$$
$$k_2 = \rho\beta_2 \frac{r^2}{4d^2}, \quad k_3 = \rho s^2 / 2, \quad \left(\frac{2d}{a}\right)^2 = \frac{\beta_1}{c_3^2}.$$

It should be also noted that the relation $s^2 + \beta_2 = 2c_2^2$ is valid for acoustic wave characteristics in such a medium.

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