EFFECT OF VELOCITY GRADIENTS ON THE ACOUSTICAL GUIDED MODES OF A STRUCTURE

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Abstract

The ultrasonic waves propagation through a plane linear elastic structure with a local and symmetrical gradient of acoustical velocities is investigated. We assume that the wavelength is smaller than the characteristic lengths of the velocities variation. Thus, the longitudinal and transversal wave equations in the inhomogeneous zone are decoupled as in an homogeneous layer and, for an exponential velocity profile, the solutions are given by a linear combination of Hankel functions. The effect of relative amplitudes of gradients and of relative thickness of the inhomogeneous layer, on the position of the cutoff-frequencies of guided modes existing in the structure, is examined. Numerical results show a shift of the cutoff-frequencies compared to the homogeneous layer, which is positive for increasing gradients and negative for decreasing gradients.

Introduction

Many articles deal with inhomogeneous fluid media such as the atmosphere and the ocean (Brekhovskikh(1), Heller(2)), but only few papers are published about ultrasonic waves propagation in an inhomogeneous linear elastic solid (Robins(4)).

Among them, a lot of papers present numerical approached methods as the Thomson-Haskell matrix (which consists to approximate an inhomogeneous layer by a series of thin homogeneous layers) and the propagator formalism (which consists to replace the differential equations describing the waves propagation in an inhomogeneous layer by a matrix relating waves amplitudes at the top and the bottom of the layer).

In this paper, we consider a linear elastic plane structure of constant density, which presents a local and symmetrical gradient of longitudinal and transversal velocities. We are interested in the effect of relative amplitudes of the gradients and of relative thickness of the inhomogenous layer, on the guided modes of the structure and particularly, on the position of cutoff-frequencies. Our study is based on the assumption that the ultrasonic wavelength is smaller than the characteristic lengths of the velocities variation. This assumption is identical to the high-frequency approximation which appears in Richards's paper(3).

In these conditions, each of longitudinal and transversal waves potentials verify a wave equation with a velocity varying with depth. It is possible to obtain analytical solutions for these equations. So, we have chosen exponential velocity profiles.

Writing the appropriate jump-conditions for the displacements and stresses at each interface, we obtain the dispersion equation of the guided modes of the structure and particularly the cutoff-frequencies one. Their position depends on the relative amplitudes of gradients and the relative thickness of the inhomogeneous zone.

1 Wave equations and study of guided modes

We consider the structure presented in Figure 1. We denote: ρ the density, c_{LP} and c_{TP} the longitudinal and transversal waves velocities in the homogeneous layers $S_{1'}$ and $S_{2'}$ and $(1-\alpha)\,l$ their thicknesses.

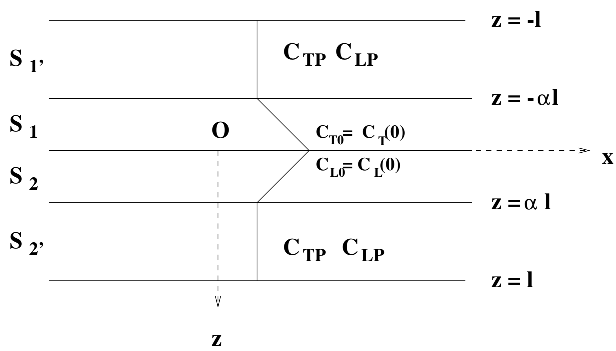

Figure 1: Geometry of the model for positive gradients

In the S_1 and S_2 layers of αl thickness ($\alpha \in [0,1]$), the velocities vary exponentially with depth:

$$c_L(z) = c_{LP}\; e^{\,a_L(l-|z|)},\; c_T(z) = c_{TP}\; e^{\,a_T(l-|z|)}.$$

Then, $c_{L0} = c_{LP}\; e^{\,a_L\, l}$ and $c_{T0} = c_{TP}\; e^{\,a_T\, l}$ are the velocities values in $z = 0$. These values are maximal if a_L and a_T are positive and minimal if a_L and a_T are negative.

We consider a problem of plane deformations, the motion being restricted to the (xOz) plane.

In the $S_{1'}$ and $S_{2'}$ layers, the longitudinal and transversal potentials satisfy the wave equations:

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - c_{LP}^2 \Delta \phi = 0, \\ \dfrac{\partial^2 \boldsymbol{\psi}}{\partial t^2} - c_{TP}^2 \Delta \boldsymbol{\psi} = 0, \end{cases} \qquad (1)$$

and the solutions are given by:

$$\begin{cases} \phi'_1 = \left[A'_{1L}\cos(k_{Lz}(z+\alpha l)) + B'_{1L}\sin(k_{Lz}(z+\alpha l))\right] \\ e^{\,i(k_x x-\omega t)} \\ \psi'_1 = \left[A'_{1T}\cos(k_{Tz}(z+\alpha l)) + B'_{1T}\sin(k_{Tz}(z+\alpha l))\right] \\ e^{\,i(k_x x-\omega t)} \end{cases} \tag{2}$$

for the $S_{1'}$ layer, and

$$\begin{cases} \phi'_2 = \left[A'_{2L}\cos(k_{Lz}(z-\alpha l)) + B'_{2L}\sin(k_{Lz}(z-\alpha l))\right] \\ e^{\,i(k_x x-\omega t)} \\ \psi'_2 = \left[A'_{2T}\cos(k_{Tz}(z-\alpha l)) + B'_{2T}\sin(k_{Tz}(z-\alpha l))\right] \\ e^{\,i(k_x x-\omega t)} \end{cases} \tag{3}$$

for the $S_{2'}$ layer.

In a recent paper (Vlasie[5]), we have proved that, at high frequencies, in the S_1 and S_2 layers, the longitudinal and transversal potentials satisfy wave equations:

$$\begin{cases} \dfrac{\partial^2\phi}{\partial t^2} - c_L^2(z)\Delta\phi = 0, \\ \dfrac{\partial^2\boldsymbol{\psi}}{\partial t^2} - c_T^2(z)\Delta\boldsymbol{\psi} = 0. \end{cases} \tag{4}$$

So, for the exponential profile previously defined, the solutions are written as follows:

$$\begin{cases} \phi_1 = [A_{1L}\ H^{(1)}_{\nu_L}(\xi_{L1}) + B_{1L}\ H^{(2)}_{\nu_L}(\xi_{L1})]e^{\,i(k_x x-\omega t)} \\ \psi_1 = [A_{1T}\ H^{(1)}_{\nu_T}(\xi_{T1}) + B_{1T}\ H^{(2)}_{\nu_T}(\xi_{T1})]e^{\,i(k_x x-\omega t)} \end{cases} \tag{5}$$

for the S_1 layer, and

$$\begin{cases} \phi_2 = [A_{2L}\ H^{(1)}_{\nu_L}(\xi_{L2}) + B_{2L}\ H^{(2)}_{\nu_L}(\xi_{L2})]e^{i(k_x x-\omega t)} \\ \psi_2 = [A_{2T}\ H^{(1)}_{\nu_T}(\xi_{T2}) + B_{2T}\ H^{(2)}_{\nu_T}(\xi_{T2})]e^{i(k_x x-\omega t)} \end{cases} \tag{6}$$

for the S_2 layer,

where $\nu_L = \dfrac{k_x}{a_L}$, $\nu_T = \dfrac{k_x}{a_T}$, $\xi_{L1} = \dfrac{\omega}{a_L c_{LP}} e^{-a_L(z+l)}$, $\xi_{T1} = \dfrac{\omega}{a_T c_{TP}} e^{-a_T(z+l)}$, $\xi_{L2} = \dfrac{\omega}{a_L c_{LP}} e^{a_L(z-l)}$, $\xi_{T2} = \dfrac{\omega}{a_T c_{TP}} e^{a_T(z-l)}$.

Writing (i) the continuity of displacements and stresses at $z = 0$ and $z = \pm\,\alpha\, l$ and (ii) the free surface conditions at $z = \pm\, l$ interfaces, we find a 16×16 system. The zero values of its determinant correspond to the guided waves in the structure.
Putting $k_x = 0$ in the 16×16 system, we obtain particular solutions which correspond to the cutoff-frequencies of the guided modes into the structure. The equations (7), (8) correspond to the cutoff-frequencies of the longitudinal waves (the transversal ones are identical provided that we replace the index L by T).

$$\begin{aligned} &\Big[H_1^{(1)}\big(\overline{\omega}\overline{\xi_L}\big)H_1^{(2)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big) \\ &-H_1^{(2)}\big(\overline{\omega}\overline{\xi_L}\big)H_1^{(1)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big)\Big]\sin((1-\alpha)\overline{\omega}n_{LP}) \\ &-\Big[H_0^{(1)}\big(\overline{\omega}\overline{\xi_L}\big)H_1^{(2)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big) \\ &-H_0^{(2)}\big(\overline{\omega}\overline{\xi_L}\big)H_1^{(1)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big)\Big]\cos((1-\alpha)\overline{\omega}n_{LP}) = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &\Big[H_1^{(1)}\big(\overline{\omega}\overline{\xi_L}\big)H_0^{(2)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big) - \\ &H_1^{(2)}\big(\overline{\omega}\overline{\xi_L}\big)H_0^{(1)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big)\Big]\sin((1-\alpha)\overline{\omega}n_{LP}) \\ &-\Big[H_0^{(1)}\big(\overline{\omega}\overline{\xi_L}\big)H_0^{(2)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big) \\ &-H_0^{(2)}\big(\overline{\omega}\overline{\xi_L}\big)H_0^{(1)}\big(\overline{\omega}\overline{\xi_L}e^{-\overline{a_L}\alpha}\big)\Big]\cos((1-\alpha)\overline{\omega}n_{LP}) = 0. \end{aligned} \tag{8}$$

In Eqs.(7), (8), we introduce the dimensionless quantities: $\overline{\omega} = k_e\, l$ which is the unknown variable (dimensionless cutoff-frequency), and the parameters: $\overline{a_L} = a_L\, l$, $\overline{\xi_L} = n_{LP}/\overline{a_L}$, $n_{LP} = c_e/c_{LP}$ ($c_e = 1500$ m/s).

<u>Particular cases</u> :

- The cases $\overline{a_L} = 0$ or $\alpha = 0$ correspond to a $2l$ thickness homogeneous layer in which the longitudinal and transversal waves propagate with the c_{LP} and c_{TP} velocities and the cutoff-frequencies are these of the Lamb modes ones (Table 1).
- The case $\alpha = 1$ corresponds to a $2l$ thickness inhomogeneous layer in which the acoustical velocities c_L and c_T present a symmetrical gradient (Tables 1 and 2).

2 Numerical applications and discussion

For the numerical applications, we have taken $c_{LP} = 2747$ m/s, $c_{TP} = 1388$ m/s and $\rho = 1191$ kg/m^3, i.e. the characteristics of polystyrene.

In the Table (1) (respectively Table (2)), we present the longitudinal cutoff-frequencies $\varpi \in (0, 20)$ for $\overline{a_L} = 0.3$ (respectively $\overline{a_L} = -0.3$) and for different values of the parameter α varying from $2l$ thickness homogeneous layer ($\alpha = 0$) to $2l$ thickness inhomogeneous layer ($\alpha = 1$). The bolded values of ϖ do not satisfy the assumption of high frequencies and consequently, they do not serve for the ultrasonic characterization of the inhomogeneous zone.

$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 1$
2.87	**2.89**	**2.95**	**3.07**	**3.43**	**3.54**
5.75	**5.75**	**5.77**	**5.88**	**6.45**	**6.65**
8.63	**8.66**	**8.80**	**8.99**	**9.76**	**10.06**
11.50	**11.51**	**11.62**	**11.93**	**12.92**	**13.32**
14.38	14.42	14.59	14.93	16.20	16.70
17.26	17.27	17.48	17.87	19.40	19.98
20.14	20.19	20.40	20.90	22.66	23.34

Table 1: Cutoff-frequencies ϖ of guided longitudinal modes for $\overline{a_L} = 0.3$.

$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
2.87	**2.80**	**2.68**	**2.54**	**2.39**	**2.32**
5.75	**5.73**	**5.62**	**5.38**	**5.08**	**4.93**
8.60	**8.46**	**8.29**	**8.00**	**7.58**	**7.35**
11.50	**11.38**	**11.07**	**10.70**	**10.16**	**9.86**
14.34	14.17	13.82	**13.32**	**12.68**	**12.30**
17.25	17.02	16.63	16.00	15.24	14.80
20.08	19.86	19.37	18.64	17.76	17.25

Table 2: Cutoff-frequencies ϖ of guided longitudinal modes for $\overline{a_L} = -0.3$.

Comparatively to the homogeneous layer ($\alpha = 0$), we observe that the cutoff-frequencies present a positive shift for $\overline{a_L} > 0$ (Table 1) and a negative shift for $\overline{a_L} < 0$ (Table 2). When $|\overline{a_L}|$ increases, the absolute value of the shift increases and this shift is larger for the high frequencies.

For a given value of $\overline{a_L}$ we observe that:

(i) for any mode, the absolute value of the shift increases when α increases;

(ii) for any α, the shift is larger for the high frequencies.

From these observations, experimental investigations should allow a measurement of the shift and consequently an evaluation of $\overline{a_L}$ and α. A shift lower than 30 kHz could not be experimentally measured because of the precision limits of the device. Then, the evaluation of these parameters is better at high frequencies range.

References

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