Regularity and irregularity in wind instruments with toneholes or bells

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Homogeneity of emission and timbre of musical instruments is a difficult issue. Intuitively it can be related to the regularity of the geometry, even if this relation is another difficult issue. The present talk aims at contribute to the discussion about the regularity of both woodwind and brass instruments. Benade published a paper in 1960 where woodwinds were modeled as periodic media with regular toneholes. In 1974, with Jansson, he compared bells of brass instruments and exponential horns. He had a particular interest in both the definition and the effects of cutoff frequencies for wind instruments. Recently we discussed a definition of acoustic regularity in the context of woodwinds. This work is first summarized, then some open questions are discussed concerning instruments with toneholes and with bells, starting with an analogy between exponential horns and periodic lattice of toneholes.

1 Introduction

Homogeneity of emission and timbre of musical instruments is a difficult issue. Intuitively it can be related to the regularity of the geometry, even if this relation is another difficult issue. The present talk aims at contribute to the discussion about the regularity of both woodwind and brass instruments. Regularity of the woodwind toneholes was recently investigated, by distinguishing two possible definitions of the (first) cutoff frequency (see Ref.[1]): the global and the local cutoff. It has been demonstrated that woodwinds can have a great acoustical regularity. The question we wish to discuss here is the following: is it possible to find similar definitions for the brass instrument bells? For that purpose, we first summarize the previous works on woodwind, then consider the case of an exponential horn, and compare it to brass instrument bells.

2 Woodwind instruments

In his paper of 1960, Benade [2] proposed to use the theory of periodic media in order to analyze the effects of a row of tone-holes of wind instruments. He discovered the existence of an important frequency, the cutoff frequency of the lattice of open holes. He evidenced this frequency by measuring the input impedance of woodwind instruments.
For a perfectly periodic lattice, the cutoff frequency is independent of the fingering. For real instruments the small variation (see Ref. [3]) of the cutoff with respect to the fingering suggests a great regularity of the tonehole lattice. However this seems to be in contradiction with the great geometrical irregularity of the holes of a clarinet. This apparent paradox was already noticed by Benade in a posthumous article [4]. Comparing 2 clarinets, one with a regular tone hole lattice and an ordinary one, he stated that the input impedances, measured at low level, "were almost identical. This was of course interesting and happy news, because it helped justify the use of formal mathematical physics for a slowly varying lattice on a geometrically quite irregular physical structure". How can we explain this

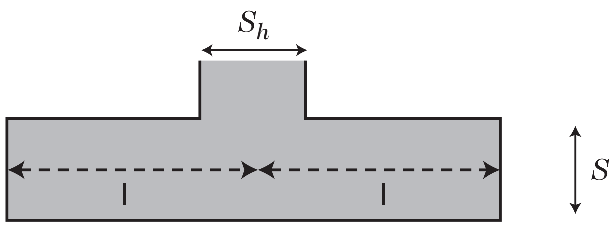

Figure 1: T-shaped cell of a regular lattice.

underlying regularity? In a recent paper (Ref. [1], see also [5]), we first remarked that for a periodic lattice, the cutoff frequency f_c is nothing else than the natural frequency of a lattice cell (see Fig. 1), with the two extremities closed by rigid surfaces. Symmetry reasons explain this feature. The following formula gives this frequency:

$$f_c = \frac{c}{2\pi\ell\sqrt{2(a/b)^2 h_t/\ell + 1/3}} \tag{1}$$

where c is the speed of sound, 2ℓ the hole spacing, a and b the radii of the tube and the hole, respectively, h_t the effective height of the hole chimney.

A logical consequence is that for an irregular lattice, it should be possible to define a "local" cutoff frequency, if it is possible to divide the lattice into cells of the same kind. This should be very interesting, because it can be proved that a succession of cells with the same cutoff has properties similar to that of a succession of identical cells (at least at rather low frequencies). Thus the paradox would be explained.

A difficulty appears with the division into cells, because the solution is not unique. Several solutions were proposed in the Ref. [1]. The simplest method avoids this difficulty: if two adjacent, symmetrical T-shaped cells *have the same natural frequency* with different lengths ℓ_1 and ℓ_2 and different hole acoustic masses $m_{h1}$and m_{h2}, Eq. (1) leads to:

$$\ell_1 m_{h1} = \ell_2 m_{h2} = \frac{\rho}{2S k_c^2}, \tag{2}$$

therefore the spacing $d = \ell_1 + \ell_2$ between the holes satisfies:

$$d = \frac{1}{k_c^2}\frac{\rho}{2S}\left(\frac{1}{m_{h_1}} + \frac{1}{m_{h2}}\right) \tag{3}$$

(ρ is the air density, S the cross section area of the tube), thus

$$f_c = \frac{c}{2\pi}\sqrt{\frac{\rho}{2S\,d}\left(\frac{1}{m_{h_1}} + \frac{1}{m_{h2}}\right)}. \tag{4}$$

This frequency is the natural frequency of a resonator of length d, with two necks corresponding to the holes with a cross section divided by 2. If it is a constant over the length of a lattice, the lattice is acoustically regular. If it is not constant, its variation can be regarded as a measure of irregularity. We can define the frequency given by Eq. (4) as a local cutoff frequency, depending in a direct way on the dimensions (masses) of two adjacent holes and their distance. This definition avoids any division of the lattice and is coherent with the definition of the global cutoff for either geometrically or acoustically perfectly regular lattices.

Fig.2 shows the results of: i) the measurements of the global cutoffs for a modern clarinet; ii) the calculation of the possible constant eigenfrequencies using a division into asymmetrical cells; iii) the calculation of the local cutoff frequencies (Eq. (4). Notice that there are two different

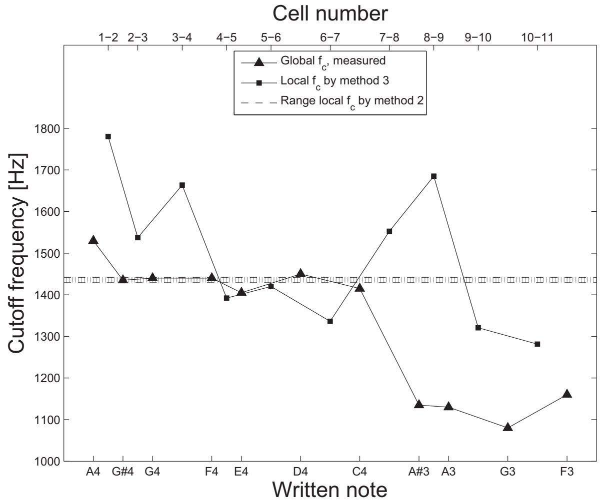

Figure 2: Comparison between experimentally determined global f_c and local f_c obtained using method 2 and 3 for the corrected model. The values of f_c for method 3 are plotted between the tones, because they are based on the acoustic mass of two subsequent equivalent tone holes. The dashed region shows the frequency range for the local f_c obtained by the division in asymmetrical cells.

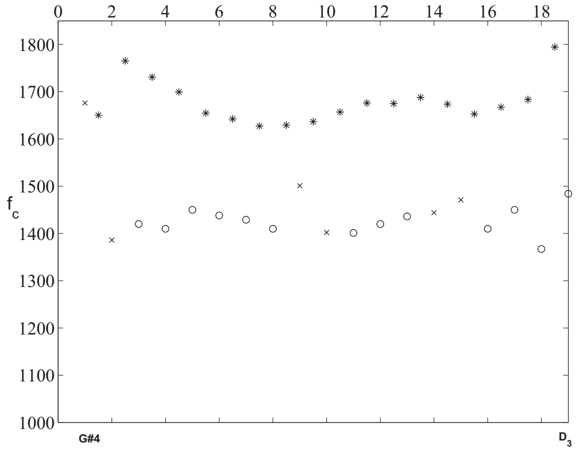

Figure 3: The local cutoff frequencies for a set of two holes are shown with stars (*). A star located at $n + 1/2$ corresponds to the cutoff frequency of the set of two holes $(n, n + 1)$. The star located at 18.5 is calculated for the hole 18 and the first vent-hole.
The circles and crosses represent the global cutoff frequencies obtained from the measurement of the input impedance, for the notes of the first register from D3 to G#4. Circles correspond to well defined values, crosses to more uncertain values.

types of axes for the abscissa: for the theoretical results, the numbers correspond to the cell numbers, while for the experimental ones, the results depend on the noted played.

Figure 2 shows an order of magnitude of the local cutoffs, in general 15% higher than the global measured values, at least at the ends of the considered register. Fig. 3 shows similar results for the prototype of "logical" clarinet [6], for which a regular increase of size and spacing is found between the 18 holes from the top to the bottom of the instrument. Again the local cutoffs are significantly higher than the global ones. The relative variations of the cutoff frequency are about 10%, while standard clarinets have a variation of the order of 40%. Therefore the computed clarinet has a satisfactory acoustical regularity of its tonehole lattice. As for a real clarinet, the mean value of the local cutoff frequencies lies around 1700 Hz. This is significantly higher than the global cutoff frequencies measured from the input impedance curve for the notes of the first register, which is around 1450 Hz (see Fig. 3).

3 Analogy between a tonehole lattice and a horn

Exponential (or catenoidal) horns are well known to have a cutoff frequency. The definition is done from the classical horn equation:

$$(pR)'' + (k^2 - R''/R)(pR) = 0. \qquad (5)$$

In this differential equation which is written in the frequency domain, p is the acoustic pressure, $R(x)$ is the radius, and $R''(x)$ is its second derivative. k is the wavenumber. The cutoff is given by $k_c^2 = R''/R$. For an exponential horn, it is a constant, and when attached to a cylindrical tube, the effect on the input impedance is the suppression of resonances above this frequency, therefore the role is similar to that of a regular tonehole lattice. In the paper [7], Benade and Jansson calculated what they called the "horn function" for brass instruments, i.e. the function $U(x) = R''(x)/R(x)$. The square root of this function can be viewed as a local cutoff frequency, because it varies with the abscissa, but we can also consider a global cutoff, by using the input impedance curve. The investigation of the analogy with tonehole lattices can therefore be interesting.

Let us consider Equation (5). For a particular type of horns, R''/R is a constant, and an evident analytical solution can be found. If $R'' > 0$, such horns are flared, of exponential or catenoid type (the function being a hyperbolic cosine); if $R'' < 0$ the horns are of the sinusoidal type, and if $R'' = 0$, we find a conical horn.

Let us study the flared horns, with positive curvature. A horn can be seen as a succession of small truncated cones whose length tends to zero. Each change in taper is equivalent to a mass in parallel, as well as an open hole (considering the simplest model for a hole): to get an analogy, the masses must be positive, and the conicity changes must be flares. Now it is easily shown that an exponential horn is the limit when the length tends to 0 for a horn consisting in truncated cones with the same length, and presenting taper changes with the same mass. Therefore the exponential horn is analogous to the limit of a cylindrical pipe drilled with identical and equidistant, open holes. This can be shown with elementary mathematics.

Now, to compare the bell of brasses to acoustically regular open hole lattices, we can compare these horns to an exponential horn. Looking at the horn equation applied to Bessel horns, defined by $R(x) = b/(x_a - x)^\nu$, where b, x_a and ν are constant, we might think of determining a *local* cutoff frequency defined as:

$$k^2 = \frac{R''}{R} = \frac{\upsilon(\upsilon+1)}{(x_a - x)^2}. \quad (6)$$

This frequency is very low at the horn entrance, because the denominator is large, and very high at the end. We intuitively understand that for this medium the regularity is qualitatively very different from that of an exponential horn. Therefore it is difficult to define a relationship between these local frequencies and the *global* cutoff frequency, that we can measure on the input impedance curve. The problem is further complicated by the fact that the models that would be useful to us are necessarily very complicated, i.e. three-dimensional. Notice that if the horn equation is written with spherical wavefronts (see Refs. [7, 8]), the horn function is strongly modified, and the difference with toneholes remain important.

The comparison between tone hole lattices and horns, that both favor the emission of higher frequencies, also gives a basic explanation of the roles of the (short) bell of a clarinet: without bell, the note for which all holes are closed may have a sound quite different from that of other notes. One way to make this note homogeneous with others is to lengthen the cylindrical pipe, and to drill other holes that will always be open: this solution is encountered in some traditional instruments. Another option is to flare the pipe termination, to form a bell which will play the same role [9], the higher frequencies being strengthened. This idea is interesting; however to our mind this simple explanation for the role of the clarinet bell needs to be further investigated.

Notice that, strictly speaking, the existence of a cutoff frequency between propagating waves and evanescent waves is a global limit property of a medium with constant characteristics, like an exponential, infinite, and lossless horn (or tonehole lattice). The only result easy to show in the case of a horn of arbitrary shape is that for the horn itself, there will be no resonance below the lowest local cutoff, corresponding to the horn entrance [10] . But this result is not of great utility.

4 Conclusion

Brass instrument bells, which are in general close to Bessel horns, have a regularity due to the property of a power law, but this regularity is very different from that of an exponential horn, thus of a woodwind. Acoustic consequences on the sound produced by these instruments remain to be investigated.

Acknowledgments

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