New lattices of sound tubes with harmonically related eigenfrequencies

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In 1994, the first author of the present paper and J. Kergomard published in acta acustica a paper entitled : *Latticies of sound tubes with harmonically related eigenfrequencies*. These lattices are made of a succession of truncated cones with equal length that have to respect certain rules. When looking at the eigenmodes with closed-open boundary conditions (i.e. reed wind instruments case), the solution were found to be stepped cone, that is cones made with a succession of cylinders whith cross sections following the law $a_n = a_1 n(n+1)/2$. At the end of the paper, the authors wrote : *Finally, it remains to be demonstrated rigorously that no other shapes of horn lattices have harmonically related eigenfrequencies*. In the present paper we show instead that other stepped horns have this property.

1 Introduction

For reed instruments the question of the harmonicity of the eigenfrequencies of the pipe is of crucial importance especially for conical reed instruments [4]. Hence the study of resonators harmonic when played with a reed is of high interest. The pipe of a reed instrument being considered as a closed open resonator, the only known possibilities considering a bore without discontinuities are the cylinder and the cone. Considering a bore with discontinuities it was shown in [3] that stepped cones made with cylinders of same length and which cross sections and are such that $a_n = (n(n+1)/2)a_1$ have also harmonic eigenfrequencies. It was conjectured at that time that this family was the only one. On another line, a new method was recently developped in [5] for computing the natural frequencies of wind instruments either with toneholes or without, relying upon a graph modelling approach [9] for solving the laplacian on networks of 1D elements. Following the theoretical presentation in [5], numerical results and other interesting properties of piecewise cylindrical resonators are presented in [6]. These first results have evidenced that other resonators actually share this property of having harmonically related natural frequencies. As a consequence, in the present work, a deeper investigation is started, with the consequence that a wide field appears to be still open to research. To begin with, one must precise what is understood here by a *harmonic resonator* : it is one for which all the natural frequencies are proportional to a given value, i.e. for which the ratio between two natural frequencies is a rational number. This is meaningful only in the lossless case with a vanishing radiation impedance. On the practical point of view, only ratios with small integers (typically less than ten) will be considered. Thus one will retain resonators for which the successive ratios to some fundamental frequency f_0 brings the series : $1, 2, 3, \ldots$ but also : $1, 3, 5, \ldots$ and so on. Of course, a special interest will be about the first series, with reference to the results from [3], but other types will be presented.

2 Studying simple cases : $N \leq 4$

Finding the natural frequencies of a piecewise cylindrical resonator is quite easy in the simple cases with one or two concatenated cylinders. These natural frequencies are the solutions of some simple equation related to the geometry of the resonator. For the most general situation with n cylinders, it is much more difficult to obtain such an equation with the usual transmission line approach : the natural frequencies are then obtained through a local numerical optimization, using the phase rotation information. In [5], thanks to a graph-based modelling of wind instruments, the determinant p_n of a certain characteristic matrix, depending on the geometry (length and cross-section) of a given piecewise cylindrical resonator, has been shown to have as zeroes the natural frequencies of the resonator. Particularizing this to resonators without toneholes in [6], a three terms recursion was given that allows to compute this determinant in a simple and efficient fashion. The following notations are in order : cylinders all have the same length L and their cross-sections are denoted a_i for the i^{th} cylinder. c is the sound velocity, ρ the air density and $k = \frac{2\pi f}{c}$ the wave number. In what follows, it will be convenient to set $x = \cos(kL)$. Then, fixing this quantity, the equations to be solved for the cross-sections are polynomial. Last, the plane wave approximation is understood. For the sake of completeness, the mentionned recursion is briefly recalled now (see [6] for more details). The determinant of interest for computing the natural frequencies of a resonator with N concatenated cylinders is computed as :

$$\left\{ \begin{array}{lcl} p_0(x) & = & 1 \\ p_1(x) & = & -a_1 x \\ p_n(x) & = & -(a_{n-1}+a_n)x p_{n-1}(x) - a_{n-1}^2 p_{n-2}(x)\ ,\ n = 2, .. \end{array} \right. \tag{1}$$

and the natural frequencies of the resonator with N concatenated cylinders are given through the solutions x of $p_N(x) = 0$.

2.1 The case $N = 1$

The case with only one cylinder allows to introduce in a simple fashion the study and to relate it to well-known results. The natural frequencies are merely the solutions of the equation :

$$x = 0 \tag{2}$$

The resonances are harmonic and correspond to the series : $\{1, 3, 5, 7, \ldots\}.f_0$ where $f_0 = \frac{c}{4L}$. The closed-open cylinder is also known as *quarter wave resonator*. On its eigenvalues, it is an impedance inverter, meaning that at the frequency f_0, the reduced input impedance is the inverse of the output impedance : $\frac{Z_e}{Z_c} = \frac{Z_c}{Z_t}$ where Z_e, Z_t are the input and output impedance respectively and $Z_c = \frac{\rho c}{a}$ is the characteristic impedance, with a the cross-section of the sole cylinder. Then for two inverters placed in series the input impedance equals the output impedance.

2.2 The case $N = 2$

When considering deux concatenated cylinders, the eigenvalues are given as the solutions of the equation :

$$x^2 = \frac{a_1}{a_1 + a_2} \tag{3}$$

where a_1, a_2 are the cross-sections of the first and second cylinder respectively (it is equivalent to Eq. (7.19), p. 259

in [2]). One concludes that the eigenvalues are harmonically related if and only if $\frac{kL}{\pi}$ is a rational number because if kL is a solution then $\pi - kL$ is also a solution, by symmetry. The chosen value for kL determines the ratio of the cross-sections. A sample of results appears in table 1 : for example, $kL = \frac{\pi}{4}$ leads to $\frac{a_2}{a_1} = 1$, which corresponds to a mere cylinder with length $2L$. In the same way, $kL = \frac{\pi}{3}$ leads to $\frac{a_2}{a_1} = 3$, which is the stepped cone [3] with increasing cross-section, whereas $kL = \frac{\pi}{6}$ leads to $\frac{a_2}{a_1} = \frac{1}{3}$, which is the stepped cone with decreasing cross-section, i.e. inverting the previous one. The cross-section ratios are not necessarily integers as $kL = \frac{\pi}{5}$ leads to $\frac{a_2}{a_1} = 5 - 2\sqrt{5}$. For each choice of x, the obtained series is different, as, for example, for $kL = \frac{\pi}{3}$ the series is $\{1, 2, 4, 5, \ldots\}.f_0$ with $f_0 = \frac{c}{6L}$ but for $kL = \frac{\pi}{6}$, the series is $\{1, 5, 7, 11, 13, \ldots\}.f_0$ with $f_0 = \frac{c}{12L}$.

Table 1: Some series for $N = 2$ and possible cross-sections

$\frac{kL}{\pi}$	series	f_0	$\frac{a_2}{a_1}$
$\frac{1}{3}$	$\{1, 2, 4, 5, 7, 8, \ldots\}$	$\frac{c}{6L}$	3
$\frac{1}{4}$	$\{1, 3, 5, 7, 9, \ldots\}$	$\frac{c}{8L}$	1
$\frac{1}{5}$	$\{1, 4, 6, 9, 11, \ldots\}$	$\frac{c}{10L}$	$5 - 2\sqrt{5}$
$\frac{1}{6}$	$\{1, 5, 7, 11, 12, \ldots\}$	$\frac{c}{12L}$	$\frac{1}{3}$
$\frac{2}{5}$	$\{2, 3, 7, 8, 12, \ldots\}$	$\frac{c}{10L}$	$\frac{5-2\sqrt{5}}{9-4\sqrt{5}}$

2.3 The case $N = 3$

As for each situation where N is odd, and in particular $N = 1$, the frequency $f = \frac{c}{4L}$ is a natural frequency whatever the length L because in that case, pairs of cylinders reproduce the terminal impedance and the situation is the same as for a unique cylinder. In mathematical terms, x is a factor of $p_N(x)$ when N is odd thus when x vanishes, p_N does too. The other natural frequencies are the solutions of the equation (see [6]) :

$$x^2 = \frac{a_1 a_2 + a_1 a_3 + a_2^2}{(a_1 + a_2)(a_2 + a_3)} \tag{4}$$

As for $N = 2$, the natural frequencies are harmonic if and only if $\frac{kL}{\pi}$ is a rational number. Hence, there is only one equation for two unknowns, reminding that a_1 can be chosen arbitrarily. The consequence is that the set of solutions is a one dimensional continuum : a_2 can be chosen arbitrarily too, provided $a_2 > \frac{a_1(1-x^2)}{x^2}$. Then a_3 is given by : $a_3 = \frac{a_2(a_1+a_2)(1-x^2)}{(a_1+a_2)x^2 - a_1}$. Notice also that if $\frac{a_2}{a_1}$ and x^2 are rational numbers, then $\frac{a_3}{a_1}$ is rational too. Figure 1 illustrates the situation when $kL = \frac{\pi}{4}$, i.e. $x^2 = 1/2$ and figure 2 shows input impedance curves for different resonators in that situation. One has the solution : $\{a_1, a_2, a_3\} = \{1, 3, 6\}.a_1$ that belongs to the family of the stepped cones in [3]. But one finds also : $\{a_1, a_2, a_3\} = \{1, 2, 6\}.a_1$. Notice that for this value $x^2 = 1/2$, when $a_1 \to a_2$, $a_3 \to \infty$ and the even natural frequency progressively disappears. In the same way, when $a_2 \to \infty$, $a_3 \to a_2$; then the odd modes tend to disappear (see figure 1). The situations in between $\{1, 2, 6\}.a_1$ and $\{1, 3, 6\}.a_1$ are of particular interest (see figure 1) : one observes a great stability of the modes with respect to a large variation in the central cross-section. On another hand, for intermediate values of a_2, the harmonicity is achieved through a small variation of a_3 : consider figure 1 plotting a_3 versus a_2, all normalized to a_1, i.e. $a_1 = 1$. At the minimum of this curve, for $\frac{a_2}{a_1} = 1 + \sqrt{2}$, one has $\frac{a_3}{a_1} = 3 + 2\sqrt{2} \approx 5.83$, i.e. a relative reduction in diameter of 1.5%. In table 2, remark the case $kL = \frac{\pi}{6}$, which is also interesting as one finds among the solutions several variations of the cylinder with length $2L$, i.e. resonators having the same series $\{1, 3, 5, 7, \ldots\}.f_0$.

Figure 1: $N = 3$, a_3/a_1 vs a_2/a_1 for $x^2 = 1/2$

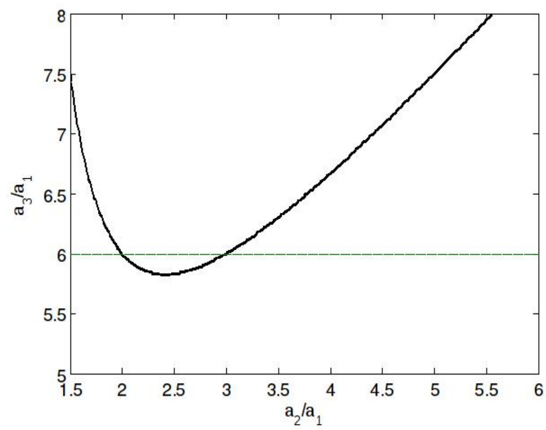
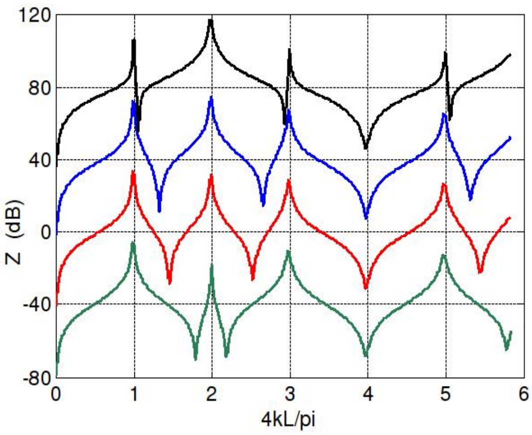


Figure 2: Input impedances, $N = 3$, $kL = \pi/4$. *Blue* : stepped cone $[1, 3, 6]$, *Red* : $[1, 2, 6]$, *Black* : $[1, 20, 22.1$, *Green* : $1, 1.1, 23.1$. Curves have been artificially shifted.



2.4 The case N=4

With four concatenated cylinders, one approaches the difficulty to find solutions with the usual transmission line method. Instead, thanks to the recursion (1), the natural frequencies are found again as the solutions of $p_4(x) = 0$, which is explicited as :

$$\begin{aligned} p_4(x) &= (a_3 + a_4)(a_2 + a_3)\, a_1 (a_1 + a_2)\, x^4 \\ &- \left((a_3 + a_4)\left((a_2 + a_3)\, a_1{}^2 + a_2{}^2 a_1\right) + a_3{}^2 a_1 (a_1 + a_2)\right) x^2 \\ &+ a_3{}^2 a_1{}^2 = 0 \end{aligned} \tag{5}$$

Table 2: Some series for $N = 3$ and a sample of cross-sections

$\frac{kL}{\pi}$	series	f_0	$\frac{a_2}{a_1}$	$\frac{a_3}{a_1}$
$\frac{1}{4}$	$\{1, 2, 3, 5, 6, 7, 9, \ldots\}$	$\frac{c}{8L}$	3	6
			2	6
			$1 + \sqrt{2}$	$3 + 2\sqrt{2}$
$\frac{1}{6}$	$\{1, 3, 5, 7, 9, \ldots\}$	$\frac{c}{12L}$	1	1
			3	1.5
$\frac{2}{6}$	$\{2, 3, 4, 8, 9, 10, 14, \ldots\}$	$\frac{c}{12L}$	6	42
$\frac{1}{5}$	$\{2, 5, 8, 12, 15, 18, \ldots\}$	$\frac{c}{20L}$	1.426	2.033

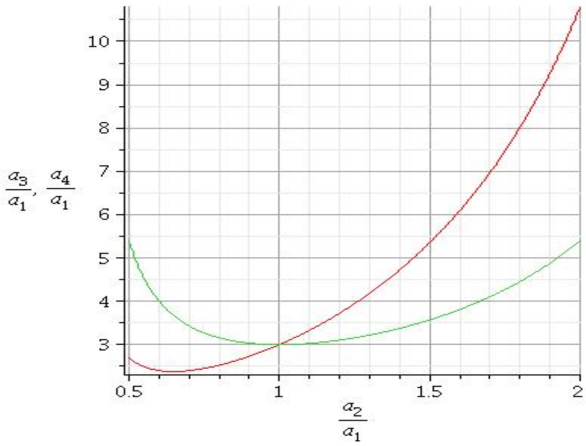
Fixing only one value for x^2 will lead to a series of resonances among which some will be harmonic of some fundamental f_0 but others will not as there are ot enough constraints. Thus this equation must be satisfied for two values of x^2, say x_1 and x_2. Both together, these will determine the series of harmonic frequencies. Remark that one cannot fix a third value for x^2 as this would lead to more equations than unknowns, because of symmetry. As before, a_1 can be taken equal to one as only the ratios to it are meaningful. Thus one is left with three parameters and two equations i.e. one degree of freedom. Choosing e.g. $a_2 > 0$ as a free parameter, one has the closed-form expressions for a_3, a_4 :

$$a_3 = -\frac{\left(\left((2a_2+a_2^2+1)x_2^2-(a_2+1)\right)x_1^2-(a_2+1)x_2^2+a_2+1\right)a_2}{\left((2a_2+a_2^2+1)x_2^2-(a_2+1)\right)x_1^2-(a_2+1)x_2^2+1} \qquad (6)$$

$$a_4 = -(1+a_2)\frac{\left(\left((1+a_2)x_2^2-1\right)x_1^2-x_2^2+1\right)\left(x_2^2-1\right)\left(x_1^2-1\right)}{\left((1+a_2)x_2^2-1\right)x_2^2\left((1+a_2)x_1^2-1\right)x_1^2} \qquad (7)$$

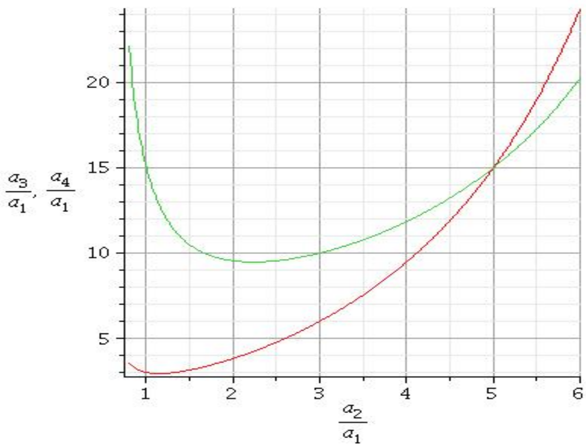
Of these relations only the positive ones are to be retained. This depends on the chosen x_1, x_2. For example, when $\frac{k_1L}{\pi} = \frac{1}{5}, \frac{k_2L}{\pi} = \frac{1}{7}$, one can check that positive solutions for a_3, a_4 are obtained only when $a_2 \in [\tan^2\pi/7, \tan^2\pi/5]$. As another example, when $\frac{k_1L}{\pi} = \frac{1}{6}, \frac{k_2L}{\pi} = \frac{1}{3}$, positive values for a_3, a_4 are obtained only when $a_2 \in [\tan^2\pi/6, \tan^2\pi/3]$. At the bounds of such intervals, the behavior of a_3, a_4 is asymptotical to the vertical line : this implies a high sensitivity of a_3, a_4 to small variations in a_2 hence a lack of robustness when thinking of design. Figure 3 shows the variations of a_3 and a_4, as functions of a_2, all normalized to a_1, i.e. with $a_1 = 1$. Outside the shown interval, both solutions are not physically relevant, being negative. Nevertheless, inside the shown interval, for a given a_2, any couple determined by intersecting both curves with a vertical line passing through this a_2 gives a resonator with harmonic frequencies. The intersection of curves means that here, for the corresponding $a_2 = 1$, $a_3 = a_4 = 3$, which actually is a resonator with length $2L$. This case appears also in fifth line of table 3 : one checks that it is coherent with the first line of table 1, but in table 3 with half the fundamental frequency f_0, as the length is twice that of table 1. As another illustration, figure 4 shows the case for which $\frac{k_1L}{\pi} = \frac{1}{5}, \frac{k_2L}{\pi} = \frac{2}{5}$, corresponding to

Figure 3: $a_3/a_1, a_4/a_1$, $N = 4, k_1L = \pi/6, k_2L = \pi/3$



the series found in [3], i.e. with the harmonic 5 lacking. Here the interval on which valid solutions may be found is $[\tan^2\pi/5, \tan^2 2\pi/5]$. Firstly, one finds back the solution $a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10$ of [3], obtained through intersecting with the vertical line $a_2 = 3$. Notice also the other integer solutions : $a_1 = 1, a_2 = 5, a_3 = 15, a_4 = 15$ and $a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 15$ for the same series. One sees that such diagrams constitute a flexible and useful

Figure 4: $a_3/a_1, a_4/a_1$, $N = 4, k_1L = \pi/5, k_2L = 2\pi/5$



tool for design. In table 3, which complements the previous diagrams, some sample series are presented with possible cross-sections realizing them. It is a remarkable fact that, in all the numerous examples that we have treated, the values of a_2 for which physically relevant solutions ($a_3 > 0, a_4 > 0$) are found are localized on intervals such as those mentionned above. Although we did not yet proved it in a formal way, we conjecture that positive solutions for a_3, a_4 exist only when $a_2 \in [\tan^2\alpha\pi, \tan^2\beta\pi]$, with $\alpha < \beta$, $\alpha, \beta \in \mathbb{Q}\cap]0, \frac{1}{2}[$, defining the chosen series through $\frac{k_1L}{\pi} = \alpha$, $\frac{k_2L}{\pi} = \beta$.

3 The general case

For N any integer, the only known complete and explicit series is the one discovered for stepped cones in [3], such that $a_N = \frac{N(N+1)}{2}a_1$ for any number N of concatenated cylinders. It was shown in that work that the $(N + 1)^{th}$ harmonic is lacking in the series for N cylinders. In the general case, one has a simple recursive mean to exhibit the equation that

Table 3: Some series for $N = 4$ and a sample of cross-sections

$\frac{k_1L}{\pi}$	$\frac{k_2L}{\pi}$	series	f_0	$\frac{a_2}{a_1}$	$\frac{a_3}{a_1}$	$\frac{a_4}{a_1}$
$\frac{1}{5}$	$\frac{2}{5}$	$1, 2, 3, 4,$ $6, 7, 8, 9, \dots$	$\frac{c}{10L}$	1	3	15
				2	3.82	9.54
				3	6	10
$\frac{1}{6}$	$\frac{1}{3}$	$1, 2, 4, 5,$ $7, 8, 10, \dots$	$\frac{c}{12L}$	0.5	2.7	5.4
				1	3	3
				1.5	5.36	3.57
$\frac{1}{7}$	$\frac{1}{5}$	$5, 7, 28, 30,$ $40, 42, \dots$	$\frac{c}{70L}$	0.25	23.16	11.34
				0.35	10.63	3.72
$\frac{1}{7}$	$\frac{1}{6}$	$6, 7, 35, 36,$ $48, 49, \dots$	$\frac{c}{84L}$	0.25	67.86	20.98
$\frac{1}{11}$	$\frac{1}{5}$	$5, 11, 44, 50,$ $60, 66, \dots$	$\frac{c}{110L}$	0.5	35.48	3.23

natural frequencies must satisfy [5, 6], although it becomes quickly even impossible to write down by hand. Using a computer algebra system with automatic code generation makes it a much easier task. Nevertheless, extending the method that was presented above for a small number of cylinders is immediate and the following procedure allows to compute, for a given number N ($N = 2n$ or $N = 2n + 1$) of cylinders, the sequence of cross-sections in order to have a certain series of harmonic natural frequencies :

1. Normalize all cross-sections to a_1, i.e. assume $a_1 = 1$
2. Compute $p_N(x)$ recursively thanks to (1)
3. If $N = 2n + 1$, then $f_0 = \frac{c}{4L}$ is a natural frequency because x is a factor of $p_N(x)$. Then set $p_N(x) := \frac{p_N(x)}{x}$, which has degree $2n$
4. Choose n values $x_i = \cos k_i L, i = 1, \dots, n$ such that $\frac{k_i L}{\pi}$ is a rational number
5. Evaluate $p_N(x_i, a_2, a_3, \dots, a_N), i = 1, \dots, n$, resulting in n polynomial equations in N (resp. $N-1$) unknowns if N is even (resp. odd). Because of symmetry, no other evaluation can be imposed.
6. Choose as parameters n among the N (resp. $N - 1$) cross-sections : $a_i, i = i_1, \dots, i_n$
7. Solve the n equations of step 5 and obtain closed-form expressions for the remaining unknown cross-sections, as functions of the free parameters $a_i, i = i_1, \dots, i_n$
8. For each choice of these set of parameters, compute the remaining n cross-sections and select the positive solutions

Diagrams such as in figures 1, 3, 4 will now depend on a number of free parameters growing with the number of cylinders of the resonator, thus one will have to use projections on planes (a_i, a_j) of chosen free parameters, in order to visualize the possible situations. Although the complexity is growing with N, they still constitute a usefull tool for design. This general situation is currently under investigation.

4 Conclusion

We have shown in this work that closed-open, piecewise cylindrical resonators with harmonically related natural frequencies constitute a very large family : for a given series of such frequencies, there exists, for $N > 2$, a continuum of resonators. In other words, this series parametrizes the family of these resonators. Inside this family, the set such that $a_N = \frac{N(N+1)}{2} a_1$ found in [3] seems to play a special role as, for the time being, it is the only one to have an explicit expression for the cross-sections, which moreover are integers, for any N. The other resonators made with the same number of cylinders and based on the same series can be considered as variant of the stepped cone. However, it must be understood that if the variant is far from the stepped cone the amplitudes of the impedance peak amplitude will be highly variable leading to resonator poorly usable for musical instruments. Cross-sections of piecewise cylindrical resonators with harmonic natural frequencies are found as the solutions of polynomial (in the a_i's) systems of equations. Among the possible solutions, a large amount is not physically relevant, being negative. The question of localizing these physically relevant solutions is thus important. From the first low dimensional cases that have been studied here, one has exhibited the real intervals where this localization occurs : for $N = 3$ (section 2.3), relevant solutions exist only when $a_2 \in [\tan(kL)^2, +\infty[$. When $N = 4$ (section 2.4), we have checked on numerous examples that physically relevant solutions exist only when $a_2 \in [\tan^2 \alpha\pi, \tan^2 \beta\pi]$ with $\alpha < \beta \in \mathbb{Q} \cap]0, \frac{1}{2}[$, making it a plausible conjecture. The generalization of this conjecture to the case of N cylinders is an exciting question, for theoretical as well as practical reasons. Future work will deal with making this conjectures established facts. We have also seen that the robustness issue when designing such resonators can be studied to a certain extent, as the sensitivity of the unknown cross-sections to small variations in the free ones : this sensitivity is very high near the bounds of the domain. This is surely useful in an optimization process, for the choice of initial values in an iterative procedure. A preliminary study in that direction is in [7]. Moreover resonators made with a large number of cylinders based on the series of a given stepped cone can be seen as a stepped cone in which the cross section is not perfectly constant. For example the series $\{1, 3, 1.5\}a_1$ can be seen as a fluctuation of the cylinder which preserve the harmonicity of the eigenfrequencies. The series $\{1, 1.5, 5.36, 3.56\}a_1$ can as well be seen as a fluctuation of the stepped cone $\{1, 3\}a_1$. This result is interesting because it can help to interpret the results of some numerical optimization of bore under constraints which lead sometimes to peculiar results [1, 8].

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