# Derivation of Nonlinear Coupling Coefficients in Internal Resonance Formulation of Cymbal Vibrations

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Timbre of cymbal greatly depends on the excitation. This phenomenon is caused by energy conversion from some natural modes to higher-order modes. This behavior is caused by geometrical nonlinearity. Therefore, the conventional electronic drum which expresses the timbre of cymbal does not have been able to achieve the low calculation cost. To solve this problem, this study aims to experimentally derive the nonlinear coupling coefficient by formulation the relationship among the mode amplitudes when cymbal is excited with sinusoidal signal for the model of cymbal considering nonlinearity. Thus we measured the vibrational distribution of cymbal using microphone array with sinusoidal excitation. The result of the experiment describes that the amplitude of higher-order modes is proportional to a square of directly excited mode amplitude. Furthermore, The nonlinear coupling coefficients are evaluated in consequence of the formulation.

# 1 Introduction

Timbre of cymbal greatly depends on the excitation strength. This phenomenon is caused by energy conversion from some natural modes to higher-order ones whose natural frequency ratio is 1 : *n* (n is an integer). This behavior is derived from geometrical nonlinearity in the thin plate vibration[1]. It is expected that the modeling technique on cymbal including nonlinearity is established for sound synthesis model of electronic musical instrument and application to new design technique of musical instruments.

Over the years a number of works describing modeling technique on cymbal including nonlinearity have been published[2-4]. The physical model using Finite Element Method (FEM) transient analysis has been published for modeling cymbal by Muenster et al[5]. The physical model which includes FEM transient analysis is beneficial because of its strict reproducibility of the cymbal vibration. At the same time, however, it is difficult to apply it to electric musical instruments due to problems of computational complexity. In the application to electrical musical instruments, it is important to solve the problem of computational complexity for reproducing the dynamic timbre change. Moreover, cymbal has great individual difference in distortion of form, natural frequency, etc., because of hand made. Thus it is expected that the result of numerical simulation contradicts timbre of practical cymbal. Therefore the model of cymbal is suited to establish by experimental method.

# 2 Sound synthesis model using multiple time scale

In the modeling on sound of musical instruments, it is not required to strictly reproduce waveform of cymbal sound. It is sufficient to reproduce the amplitude envelope of each frequency component. Therefore we are planning to apply the method of multiple scales which is used for finding an approximate solution in nonlinear dynamics[6], for the following reasons; the idea of the multiple scales method is to divide the time into some time scales such as fast periodic variation (fast time scale $t_1$) and envelope of waveform (slow time scale $t_0$) describing the decay of vibration. This idea is suitable for sound synthesis, which is mentioned above because it is sufficient to be just considered the component of slow time scale $t_0$. Therefore, we propose the nonlinear acoustical model of cymbal using multiple time scale. A schematic representation of the nonlinear acoustical model is given in **Figure 1**. Each sinusoidal generator describes natural modes in cymbal, and output corresponds to input frequency $f_k$ and input amplitude function $A_k(t_0)$. The

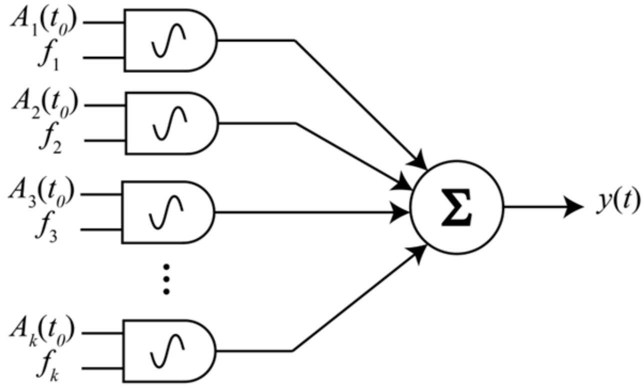


Figure 1: Time varying additive synthesis model for cymbal.

output signal reproduces the timbre of cymbal due to add the output signal from each sinusoidal generator. The output signal $y(t)$ which comes from adder is described following equation.

$$y(t) = \sum_{k=1}^{K} A_k(t_0) \cdot \sin(2\pi f_i \cdot t_1 + \varphi_k). \quad (1)$$

The condition of input signal $A_k(t_0)$ and $f_k$ is$(1 \leq k \leq K)$.

The amplitude function $A_k(t_0)$ changes depending one of other amplitude function which is caused by inner resonance. According to Fletcher et al., the amplitude function of $i$ th mode $A_i(t_0)$ which is excited by energy conversion from only fundamental mode is described[7]. The relationship between amplitude function of excited fundamental mode $A_1(t_0)$ and the amplitude function of $i$ th mode $A_i(t_0)$ whose natural frequency ratio is $n \approx f_i / f_0$, and described following equation.

$$A_1(t_0) = A_1(0) \cdot e^{-(t_0/\tau_1)}, \quad (2\text{-}1)$$

$$A_i(t_0) = \alpha_{1,i} \cdot A_1^n(t_0)\tau_i \frac{e^{-(n\,t_0/\tau_1)} - e^{-(t_0/\tau_i)}}{1-(n\tau_i/\tau_1)}. \quad (2\text{-}2)$$

Where $A_1(0)$ is amplitude of fundamental mode when $t_0 = 0$. $\tau_1$, $\tau_i$ is decay time of fundamental mode and $i$ th mode, $\alpha_{1,i}$ is nonlinear coupling coefficient between fundamental mode and $i$ th mode.

The maximum amplitude of $i$ th mode describes following equation.

$$(A_i)_{max} \propto \alpha_{1,i} \cdot A_1^n. \quad (3)$$

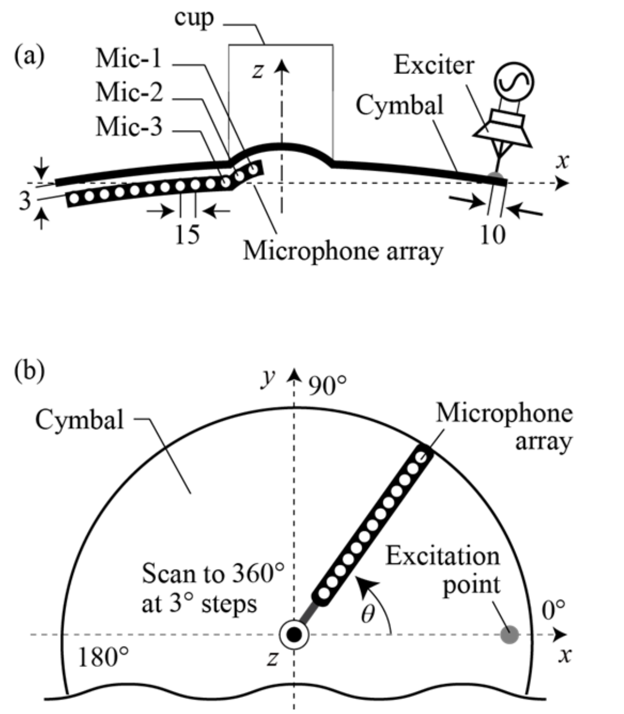


Figure 2: Arrangement of actuator and microphones. (a) Cross section view, and (b) Top view.

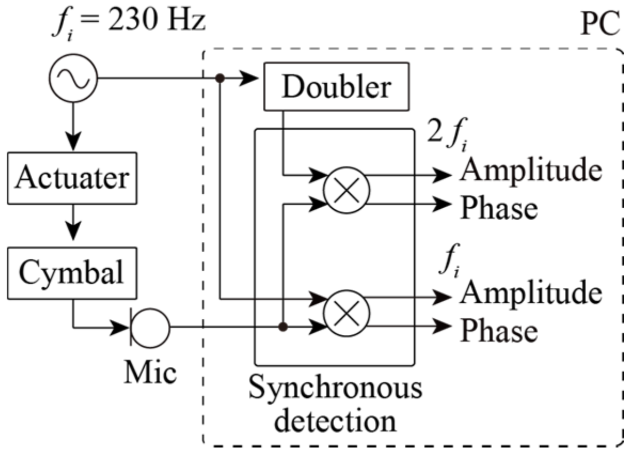


Figure 3: Data flow of Synchronous Detection for extraction of each natural mode.

It is possible that modeling of the cymbal considering nonlinearity by the amplitude function which considers energy conversion applies additive synthesis model. Additionally, due to fewer amount of computation, the sound synthesis model using multiple time scale is expected to apply sound synthesis model of electronic musical instrument.

It is necessary to fulfill the equation (2-1) and (2-2) by measuring the natural frequency $f_i$ and the decay time $\tau_i$, the nonlinear coupling coefficient $\alpha_{1,i}$ when modeling the sound synthesis model using multiple time scale. The experimental modal analysis is used to measure the natural frequency $f_i$ and the decay time $\tau_i$. However, the issue that the method of derivation of the nonlinear coupling coefficient $\alpha_{1,i}$ has not been established is raised when building the sound synthesis model using multiple time scale. Therefore we need to the method which derives the nonlinear coupling coefficient $\alpha_{1,i}$ to fulfill the equation (3). The purpose of this paper is to derive the nonlinear coupling coefficient by formulation experimentally the relationship of the mode amplitudes when cymbal is excited with sin wave.

# 3 Measurement of vibrational distribution in cymbal

The cymbal (16” AA Medium Thin Crash / SABIAN) used in experiments is 400 mm diameter, 1mm thickness, 125mm diameter of cup. The measurement system for vibrational distribution of cymbal is described in **Figure 2**. The voice coil of speaker which glued near the cymbal edge is used to excite the cymbal. The component of $f_i$ and higher harmonic component of $2f_i, 3f_i \cdots$ is generated by sinusoidal excitation at $f_i$ which is one of the natural frequency in cymbal. One of natural frequency, 230 Hz, is chosen for sinusoidal driving. The array microphones which consist of 13 elements of microphone (SPO1O3NC3-3) are located 3 mm below the cymbal surface. The vibrational distribution in cymbal is measured by rotating the microphone array at 3° intervals. The sampling duration of each step is 0.2 s, and the sampling frequency is 50 kHz. The measured voltage of microphones is converted to the vibrational amplitude in cymbal by following equation.

$$d = a \cdot \int V\,dt. \tag{4}$$

Where $d$ is temporal displacement on the surface, $V$ is the voltage measured by the microphones assumed for proportional to sound pressure, $a$ is coefficient of measured sound pressure to vibrational amplitude conversion using the laser displacement sensor (LB62 / KEYENCE).In this experiment, the coefficient $a$ is $4.01 \times 10^{-4}\,(\mathrm{m/V \cdot s})$which is calibrated. In addition, we assume that the signal from the microphone is proportional to acoustical velocity. Thus the time integration converts acoustical velocity to displacement.

**Figure 3** shows the signal processing data flow of the separation of natural modes directly excited by external force and ones excited by inner resonance. We applied the synchronous detection to extract $f_i$ component from measured signal. In addition to extract the phase information, the sinusoidal wave applied to the voice coil is used to the carrier of synchronous detection. Similar method above is applied to $2f_i$ component which is extracted using double frequency carrier. The amplitude of the driving force is 40 mN to 400 mN at 10 intervals.

# 4 Experimental results

The measured displacement distributions are shown in **Figure 4**. Where **Fig. 4(a)**, **Fig. 4(b)** indicate the $f_i$(= 230 Hz), and $2f_i$ (= 460 Hz) components, extracted by synchronous detection. The cymbal was excited at 400 mN in force amplitude. The $f_i$ component has nodes and antinodes along the circumferential direction repeating 6 times, it is called (6, 0) mode. The $2f_i$ component which is excited by inner resonance, also has nodes and antinodes repeating 9 times near the edge and 3 times near the center.

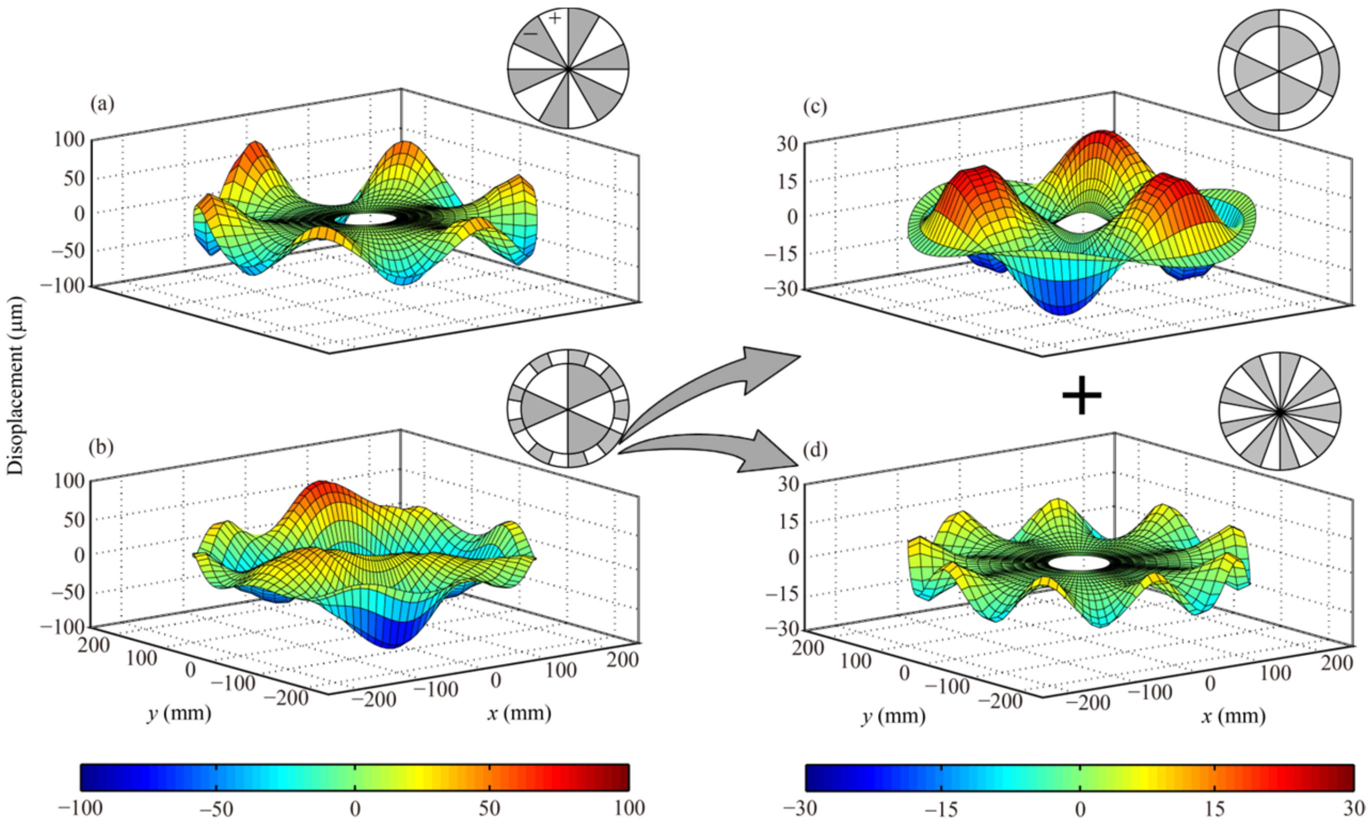


Figure 4: Measured vibrational distribution of amplitude in cymbal. (a) $f_i$ component which is excited by actuator, and (b) $2f_i$ component which is excited by inner resonance. (c) (3, 1) mode at natural frequency 455Hz. (d) (9, 0) mode at natural frequency 459 Hz.

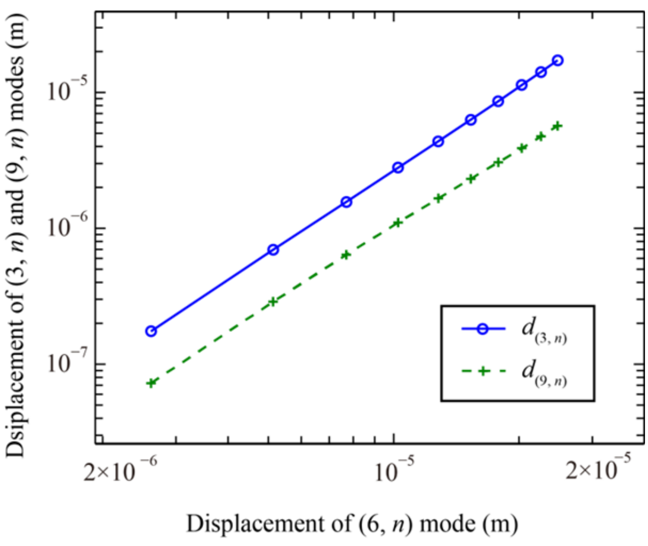


Figure 5: Strain energy of (3, 1) mode and (9, 0) mode versus strain energy of (6, 0) mode.

Thus, two natural modes are excited by inner resonance at the same time. To investigate the natural modes in cymbal which have the natural frequency in the vicinity of $2f_i$(= 460 Hz), we conducted small amplitude excitation slightly sweeping the driving frequency, so that two modes were found out; one natural mode is (3, 1) mode. The other mode is (9, 0) mode at 459 Hz. This fact, that some natural modes are simultaneously excited by inner resonance, is due to width of resonance curve. Thus two natural modes are excited at the excitation frequency, not a natural frequency of each mode. Therefore, it is difficult to separate each natural mode by synchronous detection because of two modes are both excited at $2f_i$= 460 Hz. Hence we attempt to separate this vibration into (3, 1) mode and (9, 0) mode using the orthogonality.

To divide the two natural modes whose frequency detune is small two natural modes are separated using orthogonality of mode shape[8]. The vibrational distribution of circumferential direction is expressed sinusoidal function like sin $(2\pi n)$, which form the system of orthogonal function. Thus we adopt inner product of measured distribution and the reference space frequency which has orthogonality of circumferential direction. The signal process mentioned above is applied to the result of synchronous detection at frequency $2f_i$ (= 460 Hz) as indicated in the Fig.4 (b). Each circumferential line corresponds to each microphone, we derived inner product between the sinusoidal function with 3 periods and measured distribution along circumference line of the result of $2f_i$ synchronous detection. Hence we derived amplitude of natural mode which has space frequency 3 from each circumferential line. Moreover we reproduce the mode shape of (3, 1) mode from amplitude of natural mode at each line and sinusoidal function of 3 periods. **Fig. 4(c)** describes the result of the process which mentioned above. We simultaneously applied the signal processing to the natural mode of 9 periods to circumferential direction. As a result, we obtained the displacement distribution in (9, 0) mode which is described in **Fig. 4(d)**. Thereby we obtained the vibrational distributions of (3, 1) mode and (9, 0) mode from the result of synchronous detection at identical frequency $2f_i$(= 460 Hz).

# 5 Formulation of inner resonance

**Figure** 5 shows the dependence between the amplitude of (6, 0) mode, $d_{(6,\ 0)}$, which is excited directly and the amplitudes of (3, 1) mode, $d_{(3,\ 1)}$, and (9, 0), mode $d_{(9,\ 0)}$, which are excited by inner resonance. Each amplitude ratio is plotted double-logarithmic graph with varying excitation amplitude. In addition, each amplitude of natural mode means the root mean square (RMS) of vibrational distribution. According to Fig. 5, we can observe that the plot lines have the slope, 2. Therefore the amplitudes $d_{(3,\ 1)}$ and $d_{(9,0)}$ which are excited by inner resonance are proportional to a square of amplitude $d_{(6,\ 0)}$ which is directly excited by exciter. Hence we formulated the relationship among $d_{(6,\ 0)}$ and $d_{(3,\ 1)}$, $d_{(9,\ 0)}$ . Each amplitude ratio is described by following equation,

$$d_{(3,1)} = \alpha_1 \cdot d_{(6,0)}^2, \tag{5-1}$$

$$d_{(9,0)} = \alpha_2 \cdot d_{(6,0)}^2. \tag{5-2}$$

Where $\alpha_1$ and $\alpha_2$ are called nonlinear coupling coefficient, which mean the strength of energy coupling. In addition they are constant within experimental range. We determine the value of the nonlinear coupling coefficient by experiment. Their values are $\alpha_1 = 3.0\,m^{-1}$ and $\alpha_2 = 0.8\,m^{-1}$. As seen from the equations (5-1) and (5-2), they look like the equation (3) in terms of following reason; the natural mode which has double frequency from excited one appears to be proportional to a square of directly excited mode amplitude. However, the equation (3) uses not an equal sign but a proportional mark. As we mentioned section 4, this is due to width of resonance curve. Because the natural frequency of the mode excited by inner resonance has slight deference from doubled excitation frequency $2f_i$. Thus the frequency detuning may effort to nonlinear coupling coefficients. It remains a challenge in the future research to derive the modeling technique which can determine the nonlinear coupling coefficient by not an experimental method but a theoretical or using numerical simulation method, such as finite element method (FEM). To perform this task, the parameters those contribute to nonlinear coupling coefficient like a frequency detuning will be also discussed in more detail. At present stage, our method has limitation the relationship among the amplitudes of natural modes in steady state. The transient response will be treated in the future work.

# 6 Conclusion

A major goal of our research has been to construct the sound synthesis model which is the model of cymbal considering nonlinearity. Therefore the purpose of this paper is to experimentally derive the nonlinear coupling coefficient in formulation of the relationship among the mode amplitudes when cymbal is excited with sinusoidal signal. The natural modes whose natural frequency is slightly deferent from doubled excitation frequency $2f_i$ and the natural mode which excited directly at $f_i$ are extracted from the measured vibrational distribution on the cymbal surface by the synchronous detection. Furthermore, the natural modes which are excited by inner resonance are separated into (3, 1) mode and (9, 0) mode. From this result, we evaluate the amplitude of each mode, and formulate the amplitude ratio between the directly excited mode and the modes excited by inner resonance. The nonlinear coupling coefficients are $\alpha_1 = 3.0\,m^{-1}$ , $\alpha_2 = 0.8\,m^{-1}$ in consequence of the formulation. The nonlinear coupling coefficients have been affected by the frequency detuning. It remains a task to derive the nonlinear coupling coefficient by theoretical or using numerical simulation method for constructing the proposed model. To perform this task, the parameters that contribute to the nonlinear coupling coefficient also need to be discussed in more detail. Furthermore, our method has limitation that the relationship of natural modes is in the steady state. Hence the relationship of natural modes for the transient response will be treated.

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