Measurements of Longitudinal Waves in Piano Strings and Their Dependence on Transverse String Displacement

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The existence of longitudinal waves in vibrating piano strings has been previously established, as has their importance in the characteristic sound of the piano. Previous work has established a theoretical framework describing the origins of these waves.[1] This theory indicates that longitudinal waves should appear with frequencies equal to the sum and difference frequencies of the transverse waves. Additionally, the amplitudes of the longitudinal waves should be quadratically related to the transverse displacement of the string when struck by the hammer. These predictions were tested by simultaneously measuring the power spectra of the transverse string motion and the sound produced by a piano string while being driven at two different frequencies by two independent drivers. The results indicate that longitudinal waves do appear at the sum and difference frequencies of the two driving transverse waves and that the amplitudes of these waves are linearly proportional to the amplitudes of each individual driver, in agreement with the established theory. Therefore, provided the amplitude of the transverse waves creating the longitudinal waves are linearly proportional to the initial string displacement, the amplitude of a longitudinal wave is indeed proportional to the square of the transverse displacement induced by the hammer. Measurements of the power in longitudinal waves in a piano string when the string motion is induced in the normal manner by depressing a key on the keyboard show this quadratic dependence on transverse displacement to be a good approximation.

1 Introduction

There are reports of numerous experimental investigations performed on each component of a piano, including the piano hammer, soundboard, bridges and strings. Arguably, the string motion is the most important component in creating the piano sound, therefore an understanding of the motion of a struck string is critical to produce an accurate model. Although transverse waves are known to dominate the sound produced by a piano, recent research has shown that longitudinal waves are important to the sound as well.[2]

The work reported here was undertaken in an attempt to understand how the longitudinal waves in piano strings are generated. Recently Bank and Sujbert developed a modal model describing the force on a string in the longitudinal direction.[1] Giordano and Korty experimentally confirmed a nonlinear relationship proposed by Conklin in 1999, however, they were unable to specify the power of nonlinearity.[3, 4] In the work reported here we experimentally verify a quadratic relationship between the transverse displacement and the amplitude of the longitudinal wave.

2 Theory

In 1996 Bank and Sujbert identified two types of longitudinal waves in piano strings, which they referred to as free-response and forced-response.[1] Free-response longitudinal waves occur at the frequencies of longitudinal resonances of the string. Forced-response longitudinal waves are induced by the nonlinear mixing of two transverse waves.

In 1968, Morse and Ingard published a theoretical framework describing the creation of longitudinal waves in strings induced by a transverse displacement.[5] In 2005, Bank and Sujbert applied a modal analysis to this work to predict the relationship between the amplitude of a longitudinal wave and the transverse displacement of the string. Following the derivation in ref. 1 and assuming motion in only one transverse direction, the total tension in a rigidly terminated piano string after transverse displacement can be approximated as

$$T \approx T_o + ES\left[\frac{\partial \xi}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right], \tag{1}$$

where T_0 is the tension of the string at rest, y and ξ are the transverse and longitudinal displacements respectively, and x is the position on the string. Since the longitudinal force on an element of the string is equal to the difference in tension across the string element, the force on the string in the longitudinal direction is approximately given by

$$F_x \approx \frac{\partial T}{\partial x} \approx ES\left[\frac{\partial^2 \xi}{\partial x^2} + \frac{1}{2}\frac{\partial(\partial y/\partial x)^2}{\partial x}\right]dx, \tag{2}$$

where S is the cross sectional area and E is Young's modulus.

When a piano hammer strikes the string a large number of harmonics are excited, making it difficult to identify frequencies associated with longitudinal waves. To better understand the system it is convenient to reduce the number of excited transverse frequencies to two. The transverse motion in this instance can be described by

$$y(x,t) = A_m cos(\omega_m t) sin(k_m x) + A_n cos(\omega_n t) sin(k_n x), \tag{3}$$

where A is the amplitude of the transverse modes m and n, k is the wave number, and ω is the angular frequency.

By substituting Eq. 3 into Eq. 2, it is clear that the frequencies of the longitudinal waves induced in the string are equal to the two driving frequencies as well as their sum and difference. Assuming the amplitudes of the constituent modes are equal, the force in the longitudinal direction is quadratically proportional to the transverse displacement. This proposed relationship is experimentally verified in the work reported here.

3 Experiments

Previously, researchers have used piano hammer excitation to investigate the generation of longitudinal waves in piano strings.[3, 6] However, this method complicates the identification of frequency components produced by longitudinal waves because of the overlapping transverse harmonics. By replacing the excitation of the hammer impulse with steady-state harmonic excitation it is possible to reduce the number of transverse frequencies in the string and easily identify frequencies of forced-response longitudinal waves.

To induce oscillations at only two transverse frequencies in the string, two electromagnetic shakers were used to drive either end of the piano string. Each of the shakers drove the string at a different harmonic of the transverse fundamental frequency. The amplitude of one shaker was held constant while the amplitude of the other was linearly ramped. Audio signals were recorded with a microphone near the soundboard and were used to produce a power spectrum similar to the graph shown in Fig. 1.

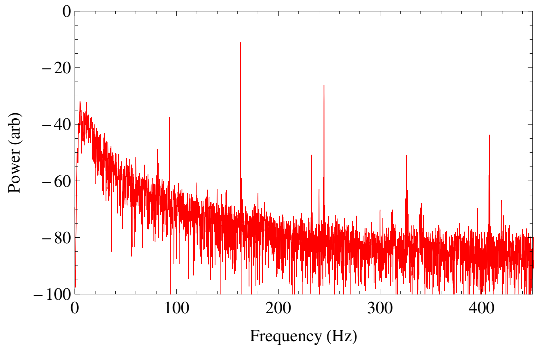

Figure 1: Power spectrum of the driven string showing significant power in the driving frequencies and the sum and difference induced by mixing.

The power spectrum shows that the two transverse driving frequencies and the sum and difference of these frequencies have significant power, which is consistent with the theory.

4 Analysis

The power in the sum and difference frequencies shown in Fig. 1 were reduced below the noise level when only one transverse frequency was excited in the string, indicating that these waves are due to mixing of transverse waves. To verify the model presented in ref. 1, the amplitude in the sum of the transverse driving frequencies was graphed as a function of transverse displacement, as shown in Fig. 2. Since the

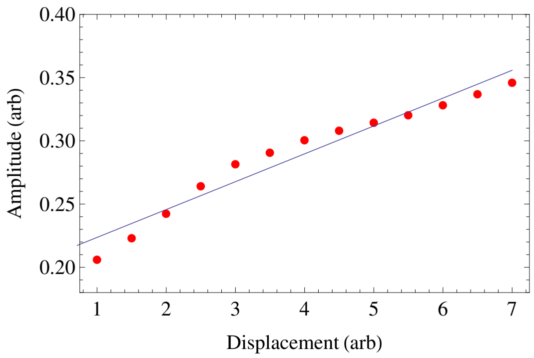

Figure 2: The amplitude of a longitudinal wave induced by nonlinear mixing as a function of transverse displacement.

amplitude of one shaker was held constant and the amplitude of the other was linearly ramped, the linear relationship in Fig. 2 implies a quadratic dependence of the power in the longitudinal waves on the transverse displacement.

In a piano, assuming the amplitudes of transverse modes are linearly proportional to initial displacement of the string caused by hammer strike, the power in the longitudinal waves is quadratically proportional to the transverse displacement. Although this experiment is not representative of the complete interactions that result from hammer excitation, previous measurements of the amplitude of longitudinal waves as a function of hammer force on the string indicates similar results.[6]

In the work reported in ref. 6, a key was depressed multiple times with various amounts of force, causing the corresponding piano hammer to strike the string and produce transverse vibrations with different amplitudes. Audio signals were recorded using a microphone and were used to calculate a power spectrum. One frequency component associated with a longitudinal wave was identified in the power spectrum and the sum of the power in the first 10 transverse harmonics was used to determine the relative transverse displacement. These results are shown in Fig. 3, where the square root of the power in the longitudinal wave is plotted as a function of the power in the transverse modes. The quadratic relationship is obvious.

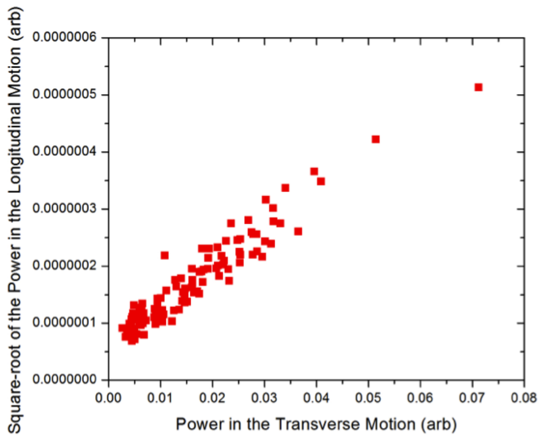

Figure 3: The square root of the power in the longitudinal direction as a function of the power in the transverse motion when the string is excited by a hammer strike.

5 Conclusions

It is clear from this work that the power in the forced-response longitudinal waves is linearly proportional to the amplitude of each of the two constituent transverse waves. By using steady-state excitation, longitudinal waves were unambiguously identified at the sum and difference frequencies as expected. Due to the small displacement of the string, we assume negligible coupling of longitudinal waves to transverse waves. However, this assumption may not be valid for hammer impulse excitation. This will be the focus of future work.

Acknowledgments

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