To what extent can a Linear Analysis predict the Behaviour of a Flute Model?

S. Terrien[a], C. Vergez[a] and B. Fabre[b]
[a]LMA - CNRS, 31, chemin Joseph Aiguier, 13009 Marseille, France
[b]LAM/d'Alembert, Sorbonne Universités, UPMC Univ. Paris 06, UMR CNRS 7190, 11, rue de Lourmel, 75015 Paris, France
terrien@lma.cnrs-mrs.fr

Although they have been widely studied for years, some aspects of the behaviour of flute-like musical instruments remain poorly understood. The study of a physical model of the instrument has demonstrated its interest in the understanding of various phenomena, such as the hysteresis related to regime changes or the variations of the frequency with the blowing pressure. As it involves both nonlinear and delayed terms, an indepth study of the state of the art flute model requires specific numerical methods, which are often computationally expensive. The simplification of the model through its linearisation around a non-oscillating trivial solution is thus particularly interesting, due to the simplicity of the calculations. The information provided by such an analysis in terms of oscillation frequency or oscillation thresholds of the different periodic solutions has been highlighted in previous work . Surprisingly enough, the present study shows that this simple linear analysis provides information about the stability zones of the different periodic solutions (i.e. the different registers), and allows to predict, in some cases, the register resulting from a transient of the mouth pressure. Such information can be obtained without solving the nonlinear equations and without computing the steady-state oscillations of the model.

1 Introduction

Various studies have highlighted the complex behaviour of flute-like instruments, and the valuable information arising from the study of a physical model of this kind of instruments [2, 5, 3]. However, even the simplest model involves both nonlinear, non smooth and delayed terms [1]. These particular features make both its time-domain integration and its analysis in terms of a (neutral) nonlinear dynamical system particularly complex, and costly in terms of computation time [3]. The drastic reduction of the model, through its linearisation around a trivial non oscillating (*i.e. static*) solution thus presents an obvious interest, due to the simplicity of calculations. It is theoretically known that this linear analysis of the model provides information about the stability properties of the considered static solution, and thus, in some cases, about the oscillation threshold (see for example [10]). Moreover, some studies have stressed the ability of this method to provide a rough approximation of the oscillation frequency and regime change thresholds [2].

This paper focuses on the information provided by the linear analysis of the state-of-the-art physical model for flute-like instruments in the case of attack transients of the mouth pressure. Such attack transients correspond to strongly non stationnary evolutions of the parameters, and to oscillations outside the vinicity of the static solution involved in the linearisation. As the linearisation involves hypothesis of both quasistatic variations of the parameters, and small perturbations around the static solution, attack transients fall, at first sight, outside the scope of the considered linear analysis. However, due to their musical importance and to the fact that they are directly controlled by the instrumentalist, their study is particularly interesting.

The state-of-the-art physical model for flute-like instruments is first presented in section 2, followed by details on its linearisation and analysis in section 3. Finally, section 4 presents the comparison between the results of linear analysis and time-domain simulations of the complete nonlinear model. Information provided by the linear analysis are discussed in terms of oscillation regime reached during an attack transient, stability ranges of the different periodic regimes, and duration and spectral nature of the transients.

2 Model for flute-like instruments

2.1 General mechanism of sound production

Although transverse flutes, recorders, or organ pipes present important differences, the general mechanism of sound production can be, in all cases, described as follows [1]. When the musician blows in the instrument, a naturally unstable jet is created at the channel exit of height h (see figure 1). This jet oscillates around the sharp edge called *labium*, which constitutes the *exciter*, providing energy to the resonator, formed by the air column contained in the pipe. The acoustic field thus created in the resonator perturbs in turn the jet at the channel exit. As the jet is naturally unstable, this perturbation is amplified while convected along the jet of length W from the channel exit to the labium, thus sustaining the oscillations of the jet around the labium. This auto-oscillation process can be represented through a feedback loop system shown in figure 2.

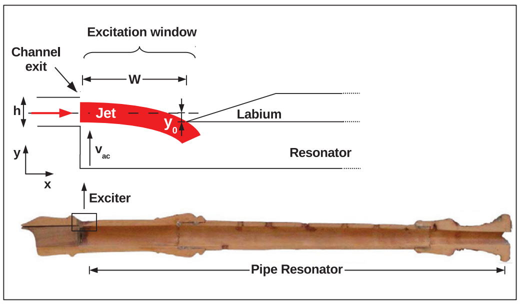

Figure 1: Recorder section and simplified representation of its exciter, constituted by the oscillation of an unstable air jet around a sharp edge called *labium*.

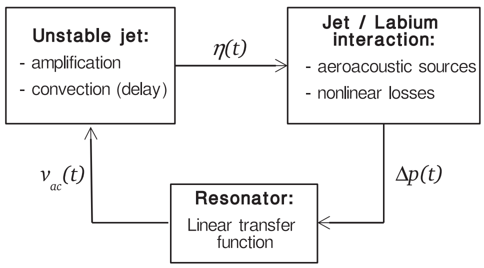

Figure 2: Basic modeling of sound production mechanism in flute-like instruments, as a feedback loop system [1].

2.2 State-of-the-art physical modeling

This mechanism of sound production is modeled through a set of equations, each of them being related to an element of the feedback loop system sketched in figure 2. The principal phenomena and the equations associated to each of these elements are briefly recalled in this section; for more details on the complete model, the reader is referred to [1, 2, 3].

Following the empirical model proposed by de la Cuadra [5], the initial transversal perturbation $\eta_0(t)$ of the jet is provided at the channel exit by the acoustic velocity $v_{ac}(t)$. Rayleigh's theory is applied to describe in a simplified way the convection and amplification of this perturbation along the jet, leading to the transversal displacement at the labium:

$$\eta(t) = \eta_0(t-\tau)e^{\alpha_i W} = \frac{h}{U_j} v_{ac}(t-\tau)e^{\alpha_i W}, \qquad (1)$$

with U_j the centerline velocity of the unstable jet (directly related to the pressure P_{mouth} in the musician's mouth, through the Bernoulli relation), and $\alpha_i \approx \frac{0.4}{h}$ an empirical amplification coefficient [5]. The delay τ, due to the convection duration, is related both to the distance W, and to the convection velocity c_v of the perturbation, with $0.3U_j < c_v < 0.5U_j$ [4, 5]. It is worth noting that τ is directly related to both the jet velocity U_j hence to the mouth pressure P_{mouth} (the higher P_{mouth}, the smaller the delay τ).

The oscillation of the jet induces an alternative flow injection on each side of the labium (inside and outside the pipe). Following the jet-drive model [6], these two localised flow sources in phase opposition, separated by a distance δ_d, constitute a dipolar pressure source $\Delta p_{src}(t)$.

Moreover, the phenomenon of vortex shedding at the labium [7] gives rise to energy loss modeled through an additional term $\Delta p_{los}(t)$ in the source equation.

The pressure source $\Delta p(t) = \Delta p_{src}(t) + \Delta p_{los}(t)$ exciting the resonator is finally written as:

$$\Delta p(t) = \frac{\rho \delta_d b U_j}{W}\frac{d}{dt}\left[\tanh\left(\frac{\eta(t)-y_0}{b}\right)\right] - \frac{\rho}{2}\left(\frac{v_{ac}(t)}{\alpha_{vc}}\right)^2 sgn((v_{ac}(t)), \qquad (2)$$

with ρ the air density, b the half width of the Bickley profile of the jet, y_0 the offset between the labium and the jet centerline (see figure 1), *sgn* the sign function, and where α_{vc} represents the *vena contracta* contraction coefficient, estimated for a sharp edge at 0.6.

The acoustical frequency response of the air column to the pressure source is given by its admittance $Y(\omega) = \frac{V_{ac}(\omega)}{\Delta P(\omega)}$, which is modelled through a modal decomposition:

$$V_{ac}(\omega) = \left[\frac{a_0}{j\omega b_0 + c_0} + \sum_{k=1}^{p}\frac{a_k j\omega}{\omega_k^2 - \omega^2 + j\omega\frac{\omega_k}{Q_k}}\right]\Delta P(\omega), \qquad (3)$$

with ω the pulsation, a_k, ω_k and Q_k the modal amplitude, the resonance pulsation and the quality factor of the k^{th} resonance mode (respectively), and a_0, b_0 and c_0 the coefficients of the so-called *uniform mode* at zero-frequency. Such coefficients are estimated, for different fingerings, from the geometrical dimensions of a Bressan Zen-On alto recorder, using the software WIAT [8].

The model is finally defined by equations 1, 2, and 3. For sake of numerical conditionning, the resulting system is made dimensionless, defining the following variables: $\tilde{t} = \omega_1 t$ and $\tilde{v}(\tilde{t}) = \frac{he^{\alpha_i W}}{bU_j} v_{ac}(t)$ (see [3] for more details).

3 Linearisation around the equilibrium

The model described in section 2 can be rewritten as a neutral delayed nonlinear dynamical system [2, 3]:

$$\dot{\mathbf{x}}(\tilde{t}) = f(\mathbf{x}(\tilde{t}), \mathbf{x}(\tilde{t}-\tilde{\tau}), \dot{\mathbf{x}}(\tilde{t}-\tilde{\tau}), \lambda). \qquad (4)$$

where λ is the set of parameters, and $\mathbf{x}$ the vector of state variables, constituted by the projections of $\tilde{v}(\tilde{t})$ on each mode of the admittance (see equation 3), and their first derivative with respect to time (see [3] for more details).

Due to its neutral delayed nature, the complete resolution and analysis of the model is particularly complex [9, 3]. Especially, time-domain simulations require very high sampling frequency (typically 1 MHz) to provide accurate results, and are thus costly in terms of computation time. The study of the corresponding linearised system around a non oscillating solution is thus particularly interesting, due to the simplicity of the calculations.

The model studied here has a trivial non oscillating solution $f(0, 0, 0) = 0$. Linearisation of system 4 around this static solution leads to the following equation:

$$\dot{\mathbf{x}}(\tilde{t}) = A_1\mathbf{x}(\tilde{t}) + A_2\mathbf{x}(\tilde{t}-\tilde{\tau}) + A_3\dot{\mathbf{x}}(\tilde{t}-\tilde{\tau}), \qquad (5)$$

where A_i denotes the partial derivative of vector function f with respect to its i^{th} argument, evaluated at the static solution 0 (see for example [10]). In this expression, terms A_2 and A_3 are directly related to the neutral nature of the system. In a more classical system of ordinary differential equations $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \lambda)$, modeling for example reed instruments, brass instruments and bowed string instruments, $A_2 = A_3 = 0$.

The stability properties of the static solution are then determined by the roots κ of the characteristic equation associated to equation 5:

$$det(\kappa I - A_1 - A_2 e^{-\kappa\tilde{\tau}} - A_3\kappa e^{-\kappa\tilde{\tau}}) = 0, \qquad (6)$$

where I represents the identity matrix [10]. In the present case, this equation is solved with the software DDE-Biftool and its extension for neutral dynamical systems [11, 9, 12]. The considered static solution remains locally stable as long as all the roots have negative real parts. Conversely, if at least one of the roots has a positive real part, any small disturbance is amplified with time, and the static solution is thus locally unstable. This analysis thus allows to determine the *Hopf bifurcation points* at which the real parts of two complex conjugate roots κ become positive. Such a Hopf bifurcation corresponds to the birth of a periodic solution (and thus to an oscillation threshold), whose frequency at this threshold is driven by the imaginary part of the considered roots. Each periodic solution is thus associated to a given instability of the static solution.

As an example, figure 3 represents real parts of the roots of equation 6, with respect to the dimensionless delay $\tilde{\tau}$ (along this paper, the roots are represented with respect to $\tilde{\tau}$ for sake of readibility and consistency with the numerical resolution method). The modal coefficients correspond to the G_4 fingering of an alto recorder. It highlights that for the range of $\tilde{\tau}$ represented, the static solution remains unstable. More precisely, two different instabilities (corresponding to two periodic solutions) exist for $0.98 < \tilde{\tau} < 1.3$. The first instability disapears for $\tilde{\tau}$ slightly below 0.2. According to the imaginary part of the root associated to this instability, it corresponds to a periodic regime at a frequency close to the first resonance frequency, the so-called *first register*. In the same way, the second instability (corresponding to the second register) disapears around $\tilde{\tau} = 0.1$. At $\tilde{\tau} = 0.98$, $\tilde{\tau} = 0.7$ and $\tilde{\tau} = 0.55$, three additional periodic solutions emerge, which respectively correspond to the third, the fourth and the fifth registers, and die out at $\tilde{\tau} = 0.05$, $\tilde{\tau} = 0.04$ and $\tilde{\tau} = 0.02$.

In addition to the prediction of oscillation thresholds and oscillation frequency at each of these thresholds, some previous studies have highlighted that such a linear analysis can provide a rough approximation of both the regime change thresholds and the oscillation frequency far from oscillation thresholds (through the imaginary parts of the different roots, not represented here) [2]. The present study focuses on the information provided by this linear analysis in the case of attack transients of the mouth pressure.

4 Results: Linear analysis and attack transients

The results of linear analysis are compared to time-domain simulations of the complete nonlinear model. Since attack transients correspond both to highly non stationary evolutions of the parameters, and to oscillations outside the vinicity of the static solution around which the system is linearised (equation 5), nothing guarantees that this method can provide valuable information.

4.1 P_{mouth} steps : arrival oscillation regime

For modal coefficients corresponding to the G_4 fingering of an alto recorder, different steps of the mouth pressure between a low value of P_{mouth} (for which the static solution is stable) and a variable target value P_t of the mouth pressure are carried out through time domain simulations of the model presented in section 2. A Runge-Kutta solver of order 3, implemented in Matlab - Simulink, has been used. Figure 4 represents, with respect to time, the mouth pressure, the acoustic velocity $v_{ac}(t)$ and the oscillation frequency, for two different values of P_t (P_t = 300 Pa and P_t = 400 Pa). The comparison of the fundamental frequencies highlights that the step with P_t = 300 Pa leads to oscillation on the first register, whereas the step with P_t = 400 Pa leads to oscillation on the second register.

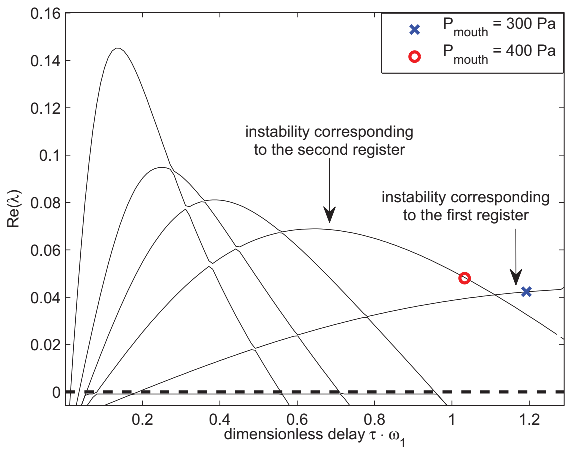

Figure 3: G_4 fingering: real parts of the roots κ (linearised model), with respect to the dimensionless delay $\tilde{\tau}$. Each (positive) *branch* of $\Re(\kappa)$ corresponds to an instability of the static solution, related to a periodic solution. The cross and the circle represent the largest value $\Re(\kappa)$, for values of P_t (300 Pa and 400 Pa) leading respectively in time-domain simulation to the first and the second register.

Examining the roots κ at these two points suggests that the arrival regime could be driven by the root with the largest

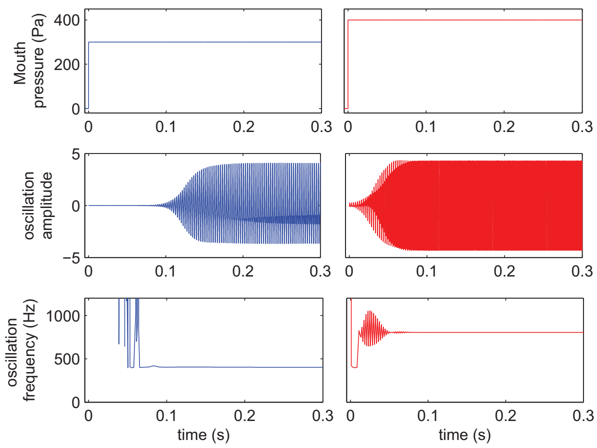

Figure 4: Time-domain simulation of the complete model, for two *steps* of P_{mouth} with a target pressure P_t = 300 Pa (left) and P_t = 400 Pa (right). Representation of the mouth pressure (high), the acoustic velocity $v_{ac}(t)$ (middle) and the fundamental frequency (bottom), with respect to time.

real part. In figure 3, which represents roots κ with respect to $\tilde{\tau}$, the blue cross and the red circle highlight the root with the largest real part, respectively for the two values of the dimensionless delay $\tilde{\tau}$ corresponding to P_t = 300 Pa and P_t = 400 Pa. In that case, the first register is obtained when the root with the largest real part corresponds to the instability giving rise to the first register, whereas the second register is obtained when the root with the largest real part is related to the existence of the second register.

This assumption is confirmed by the realisation of different steps of the mouth pressure on a wider range of P_t: as highlighted in figure 5, which represents the same kind of data as figure 3, the oscillation regime reached corresponds, in all the tested cases, to the register related to the root with the largest real part.

Through a linear analysis of the model, it is thus possible to predict the range of mouth pressure of the attacks (controlled by the musician) leading to each oscillation regime. Thus, it is possible to bring out some characteristics of a given fingering, such as for example the inhability (or at least the difficulty) to *attack* on a specific register.

4.2 Stability ranges of the periodic regimes

According to these first results, the analysis of the system linearised around its static solution allows to predict the oscillation regime resulting from a step of the mouth pressure. It thus suggests that the periodic regime related to a given instability of the static solution is necessary stable as long as the associated root is the largest one, in terms of real part. The different periodic solutions branches of the complete model and their stability properties are calculated in the software DDE-Biftool, as described in [3]. Figure 6 represents the roots κ computed for the G_4 fingering as a function of $\tilde{\tau}$ (already represented in figures 3 and 5), superimposed with the range of stability of the different corresponding periodic solution branches (*i.e.* the different registers). It highlights that for each of the five registers, if the associated root κ is the one with largest real part, the regime is stable. However the reciprocal is false.

If it does not allow to predict the precise range of stability of a given periodic regime, we conjecture that this analysis

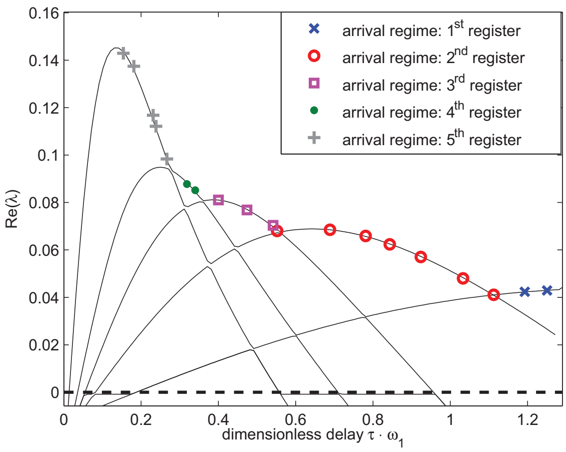

Figure 5: Lines: real parts of roots κ (linearised model), as function of the dimensionless delay $\tilde{\tau}$, for the G_4 fingering. Markers: time-domain simulation of steps of P_{mouth}; representation of the observed oscillation regime as function of the value $\tilde{\tau}_c$ corresponding to the step amplitude P_t.

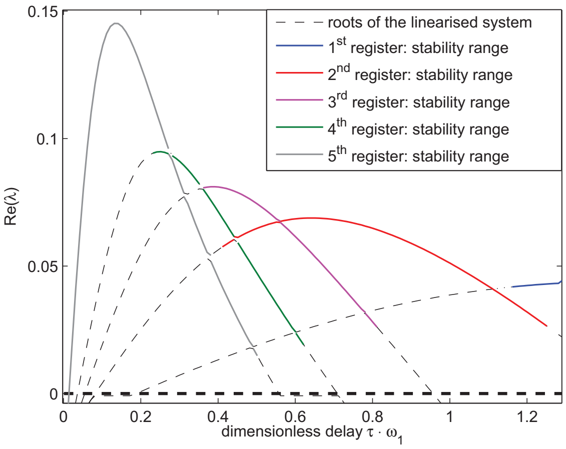

Figure 6: Dotted lines: real parts of roots κ (linearised model), as function of the dimensionless delay $\tilde{\tau}$, for the G_4 fingering. Solid lines: stability range of the periodic solution associated to each instability of the static solution (*i.e. branch* of $\Re(\kappa)$).

nevertheless provides a *minimal range of stability* for each periodic regime.

4.3 Duration and type of the transients

As real parts of roots κ characterise the amplification, with respect to time, of small disturbances superimposed on the static solution, it is expected that the duration of the transient of $v_{ac}(t)$ depends on the real part of the root associated to the arrival regime (see for example [10]). For simulations of different steps of the mouth pressure, the duration of the transient of $v_{ac}(t)$ has been compared to the largest real part of the roots κ. The duration of the transient is defined as the time between the instant at which the step of P_{mouth} occurs and the time at which the oscillation amplitude of $v_{ac}(t)$ differs by less than 10 percent from the steady-state oscillation amplitude. The results, represented in figure 7 for modal coefficients corresponding to the A_4 fingering of an alto recorder, show good agreement with the theory, in the sense that the larger the real part of the root, the shorter is the transient. However, two specific points, for $\Re(\kappa) = 0.05$ and $\Re(\kappa) = 0.07$, show unexpected behaviours.

A more careful study of this two points highlights particularly interesting phenomenon: the (long) transients of the temporal signals $v_{ac}(t)$, represented in windows (a) and (b) in figure 7, are in both cases constituted by a quasiperiodic regime. Moreover, examining the real parts of roots κ at these points shows, as highlighted in window (c) in figure 7, that in both cases, two "branches" of roots (corresponding to two different periodic solutions) intersect. Case (a) corresponds to the intersection between the branch related to the first register and the branch related to the second register. A spectral analysis of the quasiperiodic transient observed at this point highlights that its two *base frequencies* correspond to those of the 1^{st} and 2^{nd} registers.

Similarly, case (b) is located at the intersection point between the roots leading respectively to the 2^{nd} and the 4^{th} register (see (c)). As for the case (a), the spectral analysis of the quasiperiodic transient allows to identify the two base frequencies as frequencies of the 2^{nd} and 4^{th} registers.

At these points, it thus seems that the system "hesitates" between the two instabilities (leading to two different registers). As each of these instabilities involves a particular frequency, it results in a long quasiperiodic transient, constituted by these two frequencies.

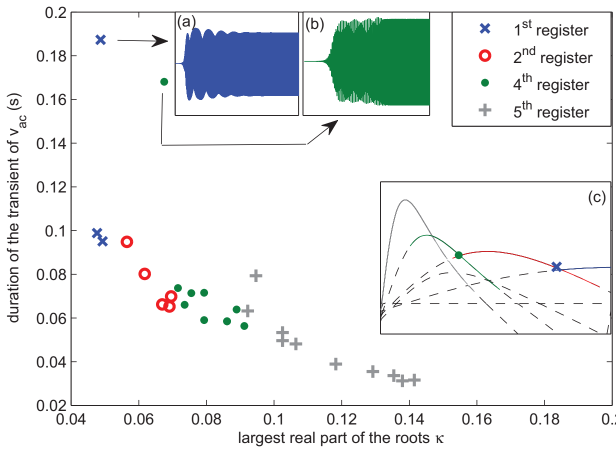

Figure 7: Time-domain simulation of mouth pressure steps, for the A_4 fingering. Transient duration of v_{ac} with respect to the largest value $\Re(\kappa)$ at the value of $\tilde{\tau}_c$ corresponding to the amplitude P_t of the step. (a) and (b): temporal signal v_{ac} with respect to time for the two pathological points. (c): $\Re(\kappa)$ of the linearised model, with respect to $\tilde{\tau}$.

It results that the computation of the roots κ not only provides an estimation of the relative durations of the transients for different values of the mouth pressure (the transient will all the shorter than the real part of the root is large), but also, in some cases, information about the nature of the transient. This result suggests that it would be possible to access, through a comparison of the real parts of the different roots, an estimation of the spectral content of the transient of v_{ac}. However, this remains to be confirmed.

4.4 Influence of the rise time of P_{mouth}

Previous sections have bring out the interesting connections between the behaviour of the complete (nonlinear) model and the roots of the linearised system, in the case of attack transients formed by *steps* of the mouth pressure. However, one can question the validity of these results in the case of more realistic transients. In a study focusing on the analysis of attack transients realised by recorder players, García [13] noted rise times of the blowing pressure between 10ms and 40ms.

To test the influence of the attack time, same kind of simulations as in section 4.1 have been achieved with linearly increasing attack transients of P_{mouth}, with rise time from 10ms to 40ms. If such profile of P_{mouth} remains a rough representation of real attack transients (mainly because of the linear increase during the attack), they are nevertheless less caricatural than the *steps* previously studied.

As in section 4.1, figure 8 presents the comparison between the oscillation regime resulting from attack transients with a rise time of 40ms (with different values of the target pressure) and real parts of the roots of the linearised system. As previously, it highlights that the arrival oscillation regime is the register resulting from the instability of the static solution associated to the root with the largest real part, and that this arrival regime can thus be predicted by the simple examination of the roots κ. Same results have been observed for different values of the rise time of P_{mouth} between 0ms and 40ms. If the influence of the attack transient profile remains to be studied, these results already argue in favor of the validity of this conjecture in the case of more realistic transients attack.

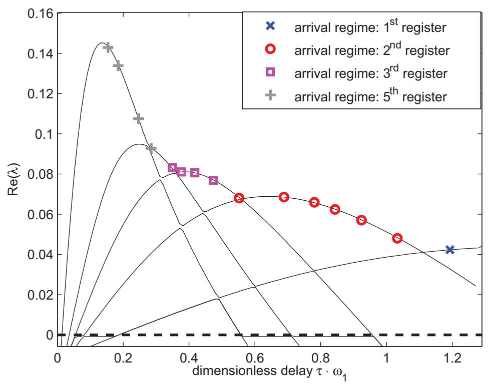

Figure 8: Lines: real parts of roots κ (linearised model), as function of $\tilde{\tau}$, for the G_4 fingering. Markers: time-domain simulation of steps of P_{mouth} with rise time of 40ms; representation of the arrival register as function of the value $\tilde{\tau}_c$ corresponding to the step amplitude P_t.

5 Conclusion

This study has presented some comparisons between the analysis of the linearised state-of-the-art model for flutes, and time-domain simulations of attack transients of the complete nonlinear model. Theoretically, attack transients fall totally out of the scope of linear analysis around the static solution. However, very surprisingly, it appears that the roots computed from the linearised model provide valuable information on attack transients of the complete nonlinear model. Indeed, the study of the real parts of the roots is enough to predict the oscillation regime resulting from a step of the mouth pressure, a *minimal range of stability* of each periodic regime, and in some cases, a qualitative estimation of the relative transients duration and spectral content. It is interesting to note that these conjectures seem to remain valid for rise times of the mouth pressure corresponding to those observed on musicians. These results thus allow, in some cases, to predict the model behaviour without solving the complete model, ans thus considerabily reducing the computation time and complexity. To the author knowledge, these surprising results have never been observed before, neither on neutral dynamical systems, nor on simpler ordinary differential systems (modelling for example reed, brass and bowed-string instruments). In order to test the validity of these conjectures for more realistic attack transients, it would be interesting to study, in addition to the rise time of the mouth pressure, the influence of its temporal profile. Indeed, only linear profiles of the mouth pressure have been considered here. Moreover, since this study suggests the existence of a relation between the quasiperiodic nature of a transient and the roots of the linearised model, it would be intereseting to explore in more depth the ability of the method to predict the spectral content of the attack transients.

References

[1] B. Fabre and A. Hirschberg, ”Physical Modeling of Flue Instruments: A Review of Lumped Models”, *Acta Acustica united with Acustica* **86**, 599-610 (2000)

[2] R. Auvray, B. Fabre and P-Y. Lagrée, ”Regime change and oscillation thresholds in recorder-like instruments”, *J. Acoust. Soc. Am.* **131**(4), 1574-1585 (2012)

[3] S. Terrien, C. Vergez, B. Fabre and D.A.W. Barton, ”Calculation of the steady-state oscillations of a flute model using the orthogonal collocation method”, *accepted for publication in Acta Acustica united with Acustica* (2014)

[4] J.W.S. Rayleigh, ”The theory of sound (second edition)”, *New York, Dover* (1894)

[5] P. de la Cuadra, ”The sound of oscillating air jets: Physics, modeling and simulation in flute-like instruments”, *PhD thesis, Stanford University* (2005)

[6] M.P. Verge, R. Caussé, B. Fabre, A. Hirschberg, A.P.J. Wijnands and A. Steenbergen, ”Jet oscillations and jet drive in recorder-like instruments”, *Acta Acustica united with Acustica* **2**, 403-419 (1994)

[7] B. Fabre, A. Hirschberg and A.P.J. Wijnands, ”Vortex Shedding in Steady Oscillation of a Flue Organ Pipe”, *Acta Acustica united with Acustica*,**82**(6), 863-877 (1996)

[8] A. Lefebvre, ”The Wind Instrument Acoustic Toolkit”, `http://www.music.mcgill.ca/caml/doku.php?id=wiat:wiat`, Accessed: 2014-03-18

[9] D. Barton, B. Krauskopf and R.E. Wilson, ”Bifurcation analysis tools for neutral delay equations: a case study”, *6th IFAC conference on Time-Delay Systems* (2006)

[10] B. Krauskopf, H.M. Osinga and J. Galan-Vioque, ”Numerical continuation methods for dynamical systems”, *Springer* (2007)

[11] K. Engelborghs, ”DDE Biftool: a Matlab package for bifurcation analysis of delay differential equations”, *Technical report, Katholieke Universiteit Leuven* (2000)

[12] D. Breda, S. Maset and R. Vermiglio, ”Pseudospectral approximation of eigenvalues of derivative operators with non-local boundary conditions”, *Applied Numerical Mathematics*, **56**(3-4), 318-331 (20006)

[13] F. Garcia, L. Vinceslas, J. Tubau and E. Maestre, ”Acquisition and study of blowing pressure profiles in recorder playing”, *Proceedings of the International Conference on New interfaces for Musical Expression*, Oslo, Norway (2011)