Numerical Simulation of Tanpura String Vibrations

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A numerical model of a tanpura string is presented, based on a recently developed, stability-preserving way of incorporating the non-smooth forces involved in the impactive distributed contact between the string and the bridge. By defining and modelling the string-bridge contact over the full length of the bridge, the simulated vibrations can be monitored through the force signals at both the bridge and the nut. As such it offers a reference model for both measurements and sound synthesis. Simulations starting from different types of initial conditions demonstrate that the model reproduces the main characteristic feature of the tanpura, namely the sustained appearance of a precursor in the force waveforms, carrying a band of overtones which decrease in frequency as the string vibrations decay. Results obtained with the numerical model are used to examine, through comparison, the effect of the bridge and of the thread on the vibrations.

1   Introduction

The tanpura is a fretless string instrument producing lively sounding drones typical of the musical cultures of the Indian subcontinent. It usually has four metal strings stretched over a resonant body connected to a long, hollow neck [1, 2]. Like various other Eastern string instruments, its specific overtone-rich sound results from the interaction of its strings with a slightly curved bridge, but with the additional feature of having a thin thread placed between the string and the bridge (see Figure 1). The musician adjusts the position of the thread - which is commonly made of silk or cotton - in search of a desired *javari*, i.e. the drone-like sound effect.

The main mechanical principles involved in the generation of a javari have been understood for some time. Raman [1] already noted in 1921 that the string vibrations of the tanpura and other Indian instruments will contain a full series of overtones regardless of where the string is plucked or mechanically held still, and attributed this phenomenon correctly to the impactive interaction between the bridge and the string. Burridge [3] provided a more detailed analysis of sitar string vibrations, identifying two main stages, both quasi-periodic. For the specific configuration of a tanpura, Valette et al. [4] showed that the placement of a thread can be considered as creating a "two-point bridge", which periodically reinforces high frequency waves that travel slightly ahead of waves of lower frequencies due to string stiffness.

Nevertheless, time-domain simulations of the tanpura and other "flat-bridge" string instruments have yet to advance to a level that allows a detailed quantitative comparison with experimental data; at the same time, physics-based sound synthesis models do not display the same level of realism as those of various other string instruments. One possible reason for this is that vibrations of a string interacting with a one-sided bridge constraint have mostly been studied and synthesised under various simplifying assumptions. For example, by defining the string-bridge collisions as fully inelastic [3, 5, 6], or completely lossless [7, 8], either of which may be a too severe simplification. In addition, string stiffness and losses are often omitted, but adding these to an existing formulation is relatively simple. The complementary extension of defining the contact as semi-elastic is less straightforward though, because modelling impactive contact with repelling forces - which are necessarily non-analytic functions of the transversal displacement of the string - generally poses considerable challenges with regard to the construction of stable, convergent time-stepping schemes [9]. While various simulation results have successfully been obtained (e.g. for the sitar [10]), provably stable formulations of this kind have yet to appear. A further indication of the need for a
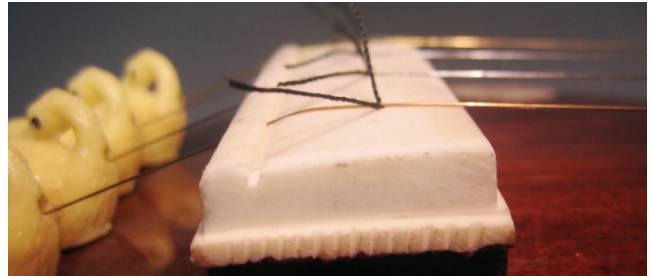

Figure 1: Side-view of a tanpura bridge.

better approach to constructing time-stepping schemes is the appearance of artefacts in the extracted signals, such as the 'spikes' visible in the nut force signals presented in [11].

This paper describes a numerical model of tanpura string vibrations, based on a recently developed energy method for modelling distributed contact in musical instruments [12]. The proposed numerical formulation is derived by discretising equations governing the transversal vibrations of a stiff string stretched over a bridge as depicted in Figure 2. While previous studies on the tanpura have defined one of the two terminations at the point where the string meets the thread (A), in our model this falls at the position of the connection with tuning bead (B). The rationale is to explicitly model contact with the thread and with the bridge over its full length, which allows the computation of the total force on the bridge. In addition, it is of help in assessing where the lower bound of the contact elasticity constants should lie.

The structure of the paper is as follows: a dissipative form of the power law defining a collision force and its discretisation is briefly discussed in Section 2. The equations governing the system of geometry (B) in Figure 2 are then presented in Section 3, followed by the formulation of the numerical scheme in Section 4. The main results are presented and discussed in Section 5.
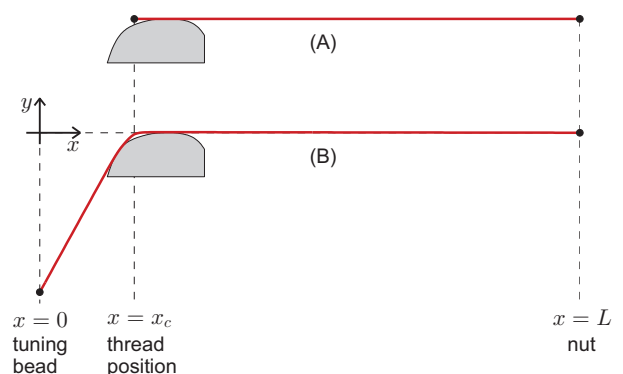

Figure 2: Tanpura string model configurations.

2 Impactive contact with damping

A well-known dissipative form of a power law for the contact force F_b between an object positioned at y with a barrier positioned below it at y_b is that by Hunt and Crossley [13]:

$$F_b = k \lfloor (y_b - y)^\alpha \rfloor \left(1 - r\frac{\partial y}{\partial t} \right), \qquad (1)$$

where α is the exponent, k is a stiffness-like term, and $\lfloor y \rfloor$ denotes $h(y) \cdot y$, where h is the Heaviside step function. The amount of damping is controlled through the coefficient r. As explained in [14], the lossless version of this power law can be discretised in stable form by first writing the force as a derivate of the potential energy, which equals

$$V(y) = \frac{k}{\alpha + 1} \lfloor (y_b - y)^{\alpha+1} \rfloor. \qquad (2)$$

This approach is readily extended to the above Hunt-Crossley form, by writing (1) as

$$F_b = -\frac{\partial V}{\partial y} + r\frac{\partial V}{\partial t}, \qquad (3)$$

and discretising using mid-point-in-time derivative approximations:

$$F_b^{n+\frac{1}{2}} = -\frac{V(y^{n+1}) - V(y^n)}{y^{n+1} - y^n} + r\frac{V(y^{n+1}) - V(y^n)}{\Delta t}, \qquad (4)$$

where n is the time index. This equation can be written into a form suitable for use in a numerical model simulating vibro-impact phenomena after subsitution of (2). In application to modelling distributed contact, as in the following section, the force term in these equations is replaced with force density, potential energy is replaced with potential energy density, and stiffness becomes stiffness per unit length.

3 Governing equations

Considering a stiff, lossy vibrating tanpura string with external force terms due to contact with the bridge (subscript 'b'), a cotton thread (subscript 'c'), and a plucking finger (subscript 'f'), the equation of motion in space-time coordinates (x, t) can be stated as

$$\rho A \frac{\partial^2 y}{\partial t^2} = \left(\tau \frac{\partial^2 y}{\partial x^2} - EI \frac{\partial^4 y}{\partial x^4} \right)\left(1 - \eta\frac{\partial}{\partial t} \right) - \rho A \gamma \frac{\partial y}{\partial t}$$
$$+ \mathcal{F}_b(x, t) + \mathcal{F}_c(x, t) + \mathcal{F}_f(x, t), \qquad (5)$$

where ρ, A, τ, E and I denote string mass density, cross-section, tension, Young's modulus, and moment of inertia, respectively, and where η and γ represent loss parameters. The respective force densities acting upon the string are, within their specific spatial domains,

$$\mathcal{F}_b(x, t) = \lfloor k_b \left[y_b(x) - y(x, t) \right] \rfloor \left(1 - r_b\frac{\partial y}{\partial t} \right), \qquad (6a)$$

$$\mathcal{F}_c(x, t) = \lfloor k_c \left[y_c(x) - y(x, t) \right] \rfloor \left(1 - r_c\frac{\partial y}{\partial t} \right), \qquad (6b)$$

$$\mathcal{F}_f(x, t) = \lfloor k_f \left[y_f(x) - y(x, t) \right] \rfloor \left(1 - r_f\frac{\partial y}{\partial t} \right), \qquad (6c)$$

and zero outside their domains. These expressions essentially represent Hunt-Crossley contact laws with unity

exponents. Some justification for the choice $\alpha = 1$ can be found in the measurement of elastic deformation of strings, such as carried out by Taguti for the silk string of the biwa [7]. A further consideration is that choosing $\alpha = 1$ with sufficiently high k values is the simplest way of avoiding unrealistically deep compression. For a string of length L, the boundary conditions are:

$$y(0, t) = y_{TB}, \qquad \left\{ \frac{\partial^2 y}{\partial t^2} \right\}_{x=0} = 0, \qquad (7a)$$

$$y(L, t) = 0, \qquad \left\{ \frac{\partial^2 y}{\partial t^2} \right\}_{x=L} = 0, \qquad (7b)$$

where y_{TB} denotes the vertical position of the left-end support (see (B) in Figure 2). The bridge profile $y_b(x)$ is taken here as measured by Guettler [15], with its maximum positioned at $x = 46.5$mm, $y = 0$mm. The terms $y_c(x)$ and $y_f(x)$ represent the shapes of the cotton thread and the plucking finger, respectively, both of which are defined as parabolic. At the start of any simulation, the initial state of the string is first solved by letting the initial vibrations due to contact forces decay. The string is then excited by setting the finger spring constant k_f to zero. Using $\mathcal{T}$ to denote the kinetic component, the system energy is:

$$H = \int_0^L \left[\mathcal{T}(p) + \mathcal{V}_t(u) + \mathcal{V}_s(v) + \mathcal{V}_b(y) + \mathcal{V}_c(y) + \mathcal{V}_f(y) \right] dx \qquad (8)$$

where the energy densities are zero outside the respective domains and otherwise

$$\mathcal{T}(p) = \frac{p^2}{2\rho A}, \qquad \mathcal{V}_t(u) = \frac{1}{2}\tau u^2,$$

$$\mathcal{V}_s(u) = \frac{1}{2}EIv^2, \qquad \mathcal{V}_b(y) = \frac{k_b}{2}\lfloor (y_b - y)^2 \rfloor, \qquad (9)$$

$$\mathcal{V}_c(y) = \frac{k_c}{2}\lfloor (y_c - y)^2 \rfloor, \qquad \mathcal{V}_f(y) = \frac{k_f}{2}\lfloor (y_f - y)^2 \rfloor,$$

and where

$$p = \rho A\frac{\partial y}{\partial t}, \qquad u = \frac{\partial y}{\partial x}, \qquad v = \frac{\partial^2 y}{\partial x^2}. \qquad (10)$$

We may now, following a path similar to that of the lossless formulation in [12], rewrite the system dynamics in terms of its energy densities and losses per unit length as follows

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{V}_\tau}{\partial u} \right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{V}_s}{\partial v} \right) - \frac{\partial \mathcal{V}_b}{\partial y} - \frac{\partial \mathcal{V}_c}{\partial y} - \frac{\partial \mathcal{V}_f}{\partial y}$$
$$- \tau\eta\frac{\partial^3 y}{\partial t\partial x^2} + EI\eta\frac{\partial^5 y}{\partial t\partial x^4} - \rho A\gamma\frac{\partial y}{\partial t}$$
$$+ r_b\frac{\partial \mathcal{V}_b}{\partial t} + r_c\frac{\partial \mathcal{V}_c}{\partial t} + +r_f\frac{\partial \mathcal{V}_f}{\partial t}, \qquad (11a)$$

$$\frac{\partial y}{\partial t} = \frac{\partial \mathcal{T}}{\partial p}. \qquad (11b)$$

The total force on the bridge is

$$F_b(t) = \int_0^L \left[\mathcal{F}_b(x, t) + \mathcal{F}_c(x, t) \right] dx. \qquad (12)$$

This expression can be used as a first approximation to the sound produced by the instrument. Another signal that is useful to extract from simulations - as it allows direct comparison with earlier studies on tanpura string vibrations [4, 2, 11] - is the force that the string exerts on the nut:

$$F_n(t) = EI\left\{ \frac{\partial^3 y}{\partial x^3} \right\}_{x=L} - \tau\left\{ \frac{\partial y}{\partial x} \right\}_{x=L}. \qquad (13)$$

4 Numerical formulation

Denoting the spatial and temporal step with Δx and Δt, respectively, the string state is discretised as $y_m^n \equiv y(m\Delta x, n\Delta t)$. Applying difference operators as in [12], the system without contact forces can be written in matrix form as

$$\boldsymbol{q}^{n+1} - \boldsymbol{q}^n = -\boldsymbol{D}\left[\left(1 + \frac{2\eta}{\Delta t}\right)\boldsymbol{y}^{n+1} + \left(1 - \frac{2\eta}{\Delta t}\right)\boldsymbol{y}^n\right]$$
$$- \frac{\gamma\Delta t}{2}\left(\boldsymbol{y}^{n+1} - \boldsymbol{y}^n\right) \quad (14a)$$

$$\boldsymbol{y}^{n+1} - \boldsymbol{y}^n = \boldsymbol{q}^{n+1} + \boldsymbol{q}^n, \quad (14b)$$

where $\boldsymbol{q}^n = \Delta t/(2\rho A)\boldsymbol{p}^n$, and where $\boldsymbol{D}$ is a square symmetric matrix representing all spatial differentiation involved [12], with numerical versions of the boundary conditions defining the elements at and near the corners. Note that in this case the condition in (7a) of a 'lowered left support' has to be incorporated. The system can be solved at each time step by finding the root $\boldsymbol{s}$ of the function

$$\boldsymbol{F} = \left[\left(1 + \frac{\gamma\Delta t}{2}\right)\boldsymbol{I} + \left(1 + \frac{2\eta}{\Delta t}\right)\boldsymbol{D}\right]\boldsymbol{s} + 2\left(\boldsymbol{D}\boldsymbol{y}^n - \boldsymbol{q}^n\right) \quad (15)$$

and subsequently updating the string displacement vector and scaled momentum vector with

$$\boldsymbol{y}^{n+1} = \boldsymbol{y}^n + \boldsymbol{s}, \quad \boldsymbol{q}^{n+1} = \boldsymbol{s} - \boldsymbol{q}^n. \quad (16)$$

Now adding contact forces to the system, force density terms at the bridge, thread, and finger are computed at separate grids for each element, using the spatial stepsizes Δx_b, Δx_c, and Δx_f, respectively. Hence a way of translating between the string grid and each of these separate grids is required. To this purpose, third-order Lagrange interpolation matrices are used as follows:

$$\bar{\boldsymbol{y}}_b^n = \boldsymbol{\mathcal{I}}_b\boldsymbol{y}^n, \quad \bar{\boldsymbol{y}}_c^n = \boldsymbol{\mathcal{I}}_c\boldsymbol{y}^n, \quad \bar{\boldsymbol{y}}_f^n = \boldsymbol{\mathcal{I}}_f\boldsymbol{y}^n. \quad (17)$$

Note that interpolation is a necessary tool in accurate positioning of the thread, the bridge, and the finger. The (scaled) contact forces are formulated at these points, for example for the bridge we have, from (4):

$$\bar{f}_{b,i}^{n+\frac{1}{2}} = -\beta_b \frac{\lfloor(y_{b,i} - \bar{y}_{b,i}^n - \bar{s}_{b,i})^2\rfloor - \lfloor(y_{b,i} - \bar{y}_{b,i}^n)^2\rfloor}{\bar{s}_{b,i}}$$
$$+ \zeta_b\left[\lfloor(y_{b,i} - \bar{y}_{b,i}^n - \bar{s}_{b,i})^2\rfloor - \lfloor(y_{b,i} - \bar{y}_{b,i}^n)^2\rfloor\right], \quad (18)$$

with $\beta_b = (k_b\Delta t^2)/(2\rho A)$ and $\zeta_b = (k_b r_b\Delta t)/(2\rho A)$, and where

$$\bar{s}_{b,i} = \bar{y}_{b,i}^{n+1} - \bar{y}_{b,i}^n. \quad (19)$$

The forces can be translated back to the string spatial coordinates using a corresponding downsampling interpolant:

$$\boldsymbol{f}_b^{n+\frac{1}{2}} = \boldsymbol{\mathcal{I}}_b^*\bar{\boldsymbol{f}}_b^{n+\frac{1}{2}}. \quad (20)$$

This can be done without affecting energy conservation for zero damping if the scaled conjugate is used as the downsampling interpolant [9]:

$$\boldsymbol{\mathcal{I}}_b^* = \left(\frac{\Delta x_b}{\Delta x}\right)\boldsymbol{\mathcal{I}}_b^t. \quad (21)$$

With the added contact forces, (15) becomes

$$\boldsymbol{F} = \left[\left(1 + \frac{\gamma\Delta t}{2}\right)\boldsymbol{I} + \left(1 + \frac{2\eta}{\Delta t}\right)\boldsymbol{D}\right]\boldsymbol{s} + 2\left(\boldsymbol{D}\boldsymbol{y}^n - \boldsymbol{q}^n\right)$$
$$- \boldsymbol{f}_b^{n+\frac{1}{2}} - \boldsymbol{f}_c^{n+\frac{1}{2}} - \boldsymbol{f}_f^{n+\frac{1}{2}}. \quad (22)$$

where the added force terms are non-linear functions of $\boldsymbol{s}$, and where the respective interpolated versions of $\boldsymbol{s}$ are computed in the same way as for $\boldsymbol{y}$:

$$\bar{\boldsymbol{s}}_b = \boldsymbol{\mathcal{I}}_b\boldsymbol{s}, \quad \bar{\boldsymbol{s}}_c = \boldsymbol{\mathcal{I}}_c\boldsymbol{s}, \quad \bar{\boldsymbol{s}}_f = \boldsymbol{\mathcal{I}}_f\boldsymbol{s}. \quad (23)$$

Equation (22) can be solved at each time step with the multi-dimensional Newton-Rhapson method, using the Jacobian

$$\boldsymbol{J} = \left[\left(1 + \frac{\gamma\Delta t}{2}\right)\boldsymbol{I} + \left(1 + \frac{2\eta}{\Delta t}\right)\boldsymbol{D}\right] + \boldsymbol{\mathcal{I}}_b^*\boldsymbol{C}_b\boldsymbol{\mathcal{I}}_b + \boldsymbol{\mathcal{I}}_c^*\boldsymbol{C}_c\boldsymbol{\mathcal{I}}_c + \boldsymbol{\mathcal{I}}_f^*\boldsymbol{C}_f\boldsymbol{\mathcal{I}}_f, \quad (24)$$

where $\boldsymbol{C}_b$, $\boldsymbol{C}_c$, and $\boldsymbol{C}_f$ are diagonal matrices with elements

$$\{c_{b,i,i}\} = \frac{\partial \bar{f}_{b,i}^{n+\frac{1}{2}}}{\partial \bar{s}_{b,i}}, \quad \{c_{c,j,j}\} = \frac{\partial \bar{f}_{c,j}^{n+\frac{1}{2}}}{\partial \bar{s}_{c,j}}, \quad \{c_{f,k,k}\} = \frac{\partial \bar{f}_{f,k}^{n+\frac{1}{2}}}{\partial \bar{s}_{f,k}}. \quad (25)$$

5 Simulation results

To obtain representative string parameters, the diameter ($d = 0.3$ mm) and length ($L = 668$ mm) of the third string of a small travelling tanpura were measured. Taking into account the fundamental frequency of the speaking length of the string as well as the mass density and Young's modulus of steel, the tension and stiffness terms were set accordingly to $\tau = 31.47$ N m^{-1} and $EI = 8.35 \times 10^{-5}$ N m^2, with $\rho A = 5.58 \times 10^{-4}$ Kg m^{-1}. The damping parameters were set to $\gamma = 0.6$ s^{-1} and $\eta = 7 \times 10^{-9}$ s, which results in a frequency-dependent decay pattern which approximately matches that observed when the string is left in free vibration (i.e. without string-bridge interaction). The bridge and thread contact elasticity coefficients are chosen as $k_b = k_c = 1 \times 10^8$ N m^{-2}, which ensures that the effective compression does not exceed 5% of the string diameter. The contact damping coefficients are set to $r_b = r_c = 0.1$ and and $r_f = 1$. The numerical parameters are as follows: $\Delta t = 1/176.4$ ms, $\Delta x = 3\frac{1}{3}$ mm, $\Delta x_b = 0.18$ mm, $\Delta x_c = 0.15$ mm, and $\Delta x_f = 1$ mm.

5.1 Quasi-Helmholz motion

Figure 3 shows snapshots of the string motion for an initial condition that matches the shape of the first mode of the string; this is achieved by re-defining the shape of the finger. In the plots, the more recent states are represented by colour-intensive curves, with the colour-tone fading out for the earlier string states. It can be observed that the bridge collisions force the string to gradually take on a more triangular shape, indicating the excitation of the other modes of vibration. A Helmholz-like motion emerges, as illustrated by the appearance of a kink that travels along the string, as indicated by the arrows in Figure 3(c). Similar findings have been presented in [11] and in studies of various other string-bridge configurations [3, 8].
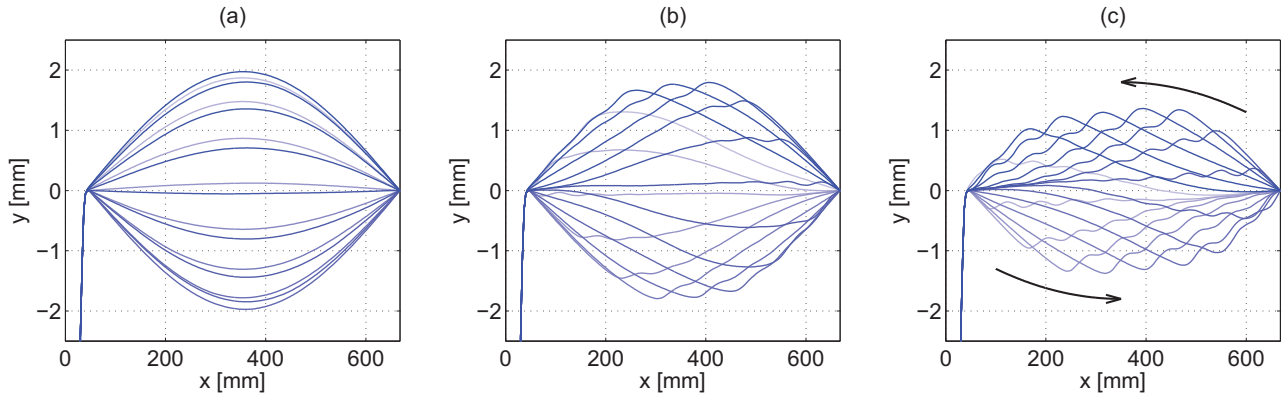
Figure 3: Snapshots of the string vibration, with the first mode initial condition, for the 1st (a), 17th (b), and 33rd (c) period of oscillation. The arrows in (c) indicate the movement of the kink, indicative of a Helmholtz-like motion.
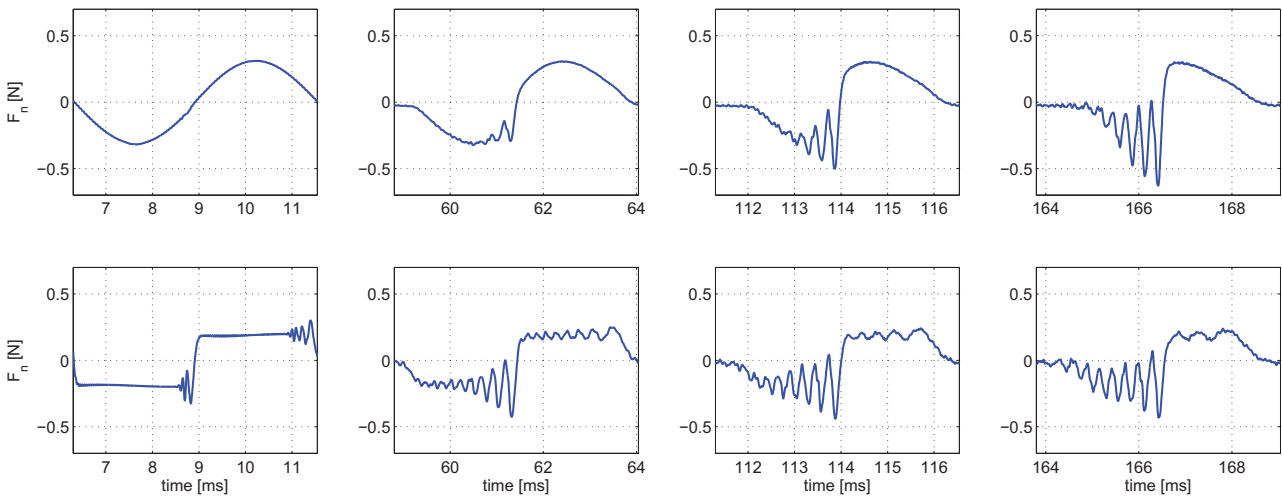

Figure 4: Evolution of the nut force signal for two different initial conditions: top: first mode shape, bottom: finger pluck shape.
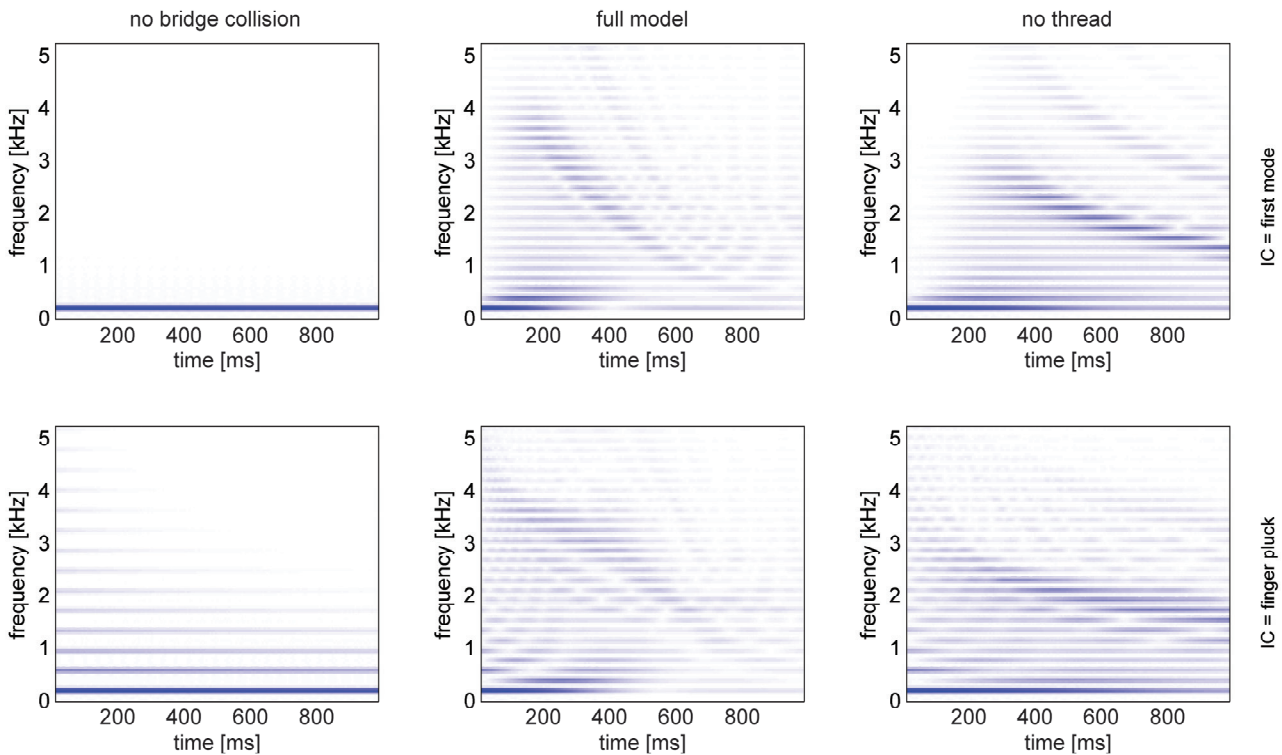

Figure 5: Spectral evolution for different initial conditions and bridge configurations.

5.2 Precursive wave generation

Figure 4 plots instances of the nut force signal for the 'first mode shape' and the 'finger pluck' cases. For both initial conditions, a gradual development toward having one sustained precursor per cycle is observed. As explained in [4], this precursor is a packet of high-frequencies arriving at the nut before the lower frequencies due to the string stiffness. The precursor keeps being 'fed' once per cycle with high-frequency components through the non-linear interaction at the bridge end, which destroys the symmetry of the force waveform periods. In the case of the plucked finger, the precursor that originally appears in the other half of the waveform gradually fades out due to frequency-dependent losses.

5.3 Influence of the bridge and the thread

Figure 5 shows the spectral evolution of the nut force signal obtained from the simulation for both initial conditions, and for three different bridge configurations. For the two plots on the left, the bridge stiffness per unit length k_b is set to zero for $x > x_c$, where $x_c = 40$mm is the position of the the thread. This means that the speaking length of the string is free to vibrate, resulting in independently decaying modes, with no energy conversion from the fundamental to any of the other modes.

With the bridge and thread in place (i.e. the full model), all modes are excited for both initial conditions, and the precursor can be observed as a formant region with a spectral centroid that varies over time. For the plucked finger case, at first the formant frequency decays, then briefly stays approximately constant, followed by a period of slower decay. The appearance of these distinct regimes in the formant centre frequency pattern is in accordance with the analysis of experimentally obtained nut force signals in [4].

Finally, the two plots on the right-hand side show the spectral evolution when no cotton thread is present in the system. In the real instrument, the removal of the thread or even a small adjustment of its position will result in a much reduced formant/precursor effect, with the sound being more similar to that of a string freely vibrating (i.e with no impactive bridge interaction). However, as seen in the plots, the model still produces vibrations with a strong formant-like feature in the spectrum. A possible explanation is that - unlike in the presented model - the vibrations of a real string are not restricted to the vertical plane. That is, a level of coupling between the two transversal polarisations, either at end points or through distributed non-linear coupling (i.e. 'string whirling') is common to all string instruments. The vertical string motion does not contribute to collisions with the bridge, so with the added plane of motion, there may be more tight conditions for triggering a strong javari. In the light of this notion it is worthwhile noting that Raman's studies include photographic evidence of the string deflection being equally prominent in both planes for string vibrations with a javari [16].

6 Conclusions

A numerical model for simulation of tanpura string vibrations has been presented. The results generated with the simulations are qualitatively in agreement with measurements and findings from earlier studies, but they also reveal that the model fails to predict the effect of removing the thread. Further research could establish which refinements and extensions would be needed to address this discrepancy. Examples of nut and bridge force signals are supplied in audio format on the accompanying website (`www.somasa.qub.ac.uk/~mvanwalstijn/isma14`).

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