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A CLOSED FORMULATION OF SOUND FIELDS IN FRONT OF PERIODICAL ABSORBENTS

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ABSTRACT
Excessive back scattering from periodic resonant absorbers has been modelled in a couple of ways in the wave number domain. A drawback with the available approaches is that the corresponding equation matrices are infinite and therefore have to be truncated in order to obtain a solution. In this paper the approach is based on a periodical assumption and yields a closed formulation that can be solved in the wave domain without truncation. It is possible to include an optional field at the backside of a periodic absorber. Furthermore, it is possible to transform this solution back to the spatial domain and obtain an exact formulation. The infinite series can then be expressed in closed form by utilising geometrical series.

1 - INTRODUCTION
The effect on the sound field in front of resonant absorbers caused by periodicity of the structure, has principally been modelled in two ways, both of which in the wave number domain. These models are designed to account for the excessive back scattering which occur at frequencies above a limit, governed by the proportions of the periodicity. Mechel modelled this phenomenon by expressing the pressure field as a series of pressure amplitude components (Hartree Harmonics) in the wave number domain and then solving the corresponding equation [1]. Takahashi [2] used a formulation similar to approaches used on scattering problems in optics. Both of these approaches yield an equation system consisting of an infinite set of components. The convergence rate for increasing number of included components is fast, but these are nevertheless unsatisfactory approaches because the equation matrices have to be truncated. However, it is possible to make use of an approach that has been used in e. g. structural acoustics by Mace in his analysis of periodically stiffened fluid-loaded plates [3] and [4]. This approach is based on Floquet's principle, i.e. a periodical assumption, which combined with use of the Poisson summation formula yields a closed formulation. Since the truncation can be suspended until the formulation is transformed back to the spatial domain, the physical interpretation is more straightforward. But an advantage with even more potential is that the resulting sums in several cases can be given a closed formulation.

2 - FORMULATION OF THE PROBLEM
In the following cases the basic ideas from Mace's work are utilised. The first instance is the simple case of infinite periodic slits. Consider an infinite plate lying in the plane y=0 (positive direction outward from the surface) with infinite slits along the lines x=nl, in the z-direction, where n is integral. The plate is backed by an air gap with depth d. The dependence of t, z are the same for all components and thus the corresponding time dependence and any component in the z-direction is suppressed through this work. The absorber is excited by a pressure which can be written as:

$$\underline{p}_{in}\left(x,y,z,t\right)=p_0e^{-i(k_xx+k_yy+k_zz-\omega t)}\;\Rightarrow\;\underline{p}_{in}\left(x,y\right)=p_0e^{-i(k_xx+k_yy)} \tag{1}$$

due to the periodicity the pressure satisfy the periodic relationship (Floquet's principle)

$$p\left(x+l,y\right)=p\left(x,y\right)e^{-ik_xl} \tag{2}$$

where the wave components are related by (angles according to standard spheric coordinates)

$$\left\{ \begin{array}{l} k_x = k\sin\theta\cos\varphi \\ k_y = k\cos\theta \\ k_z = k\sin\theta\sin\varphi \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} k_x = k\sin\theta\cos\varphi \\ k_y = k\cos\theta \end{array} \right. \tag{3}$$

The resulting scattered field from the absorbent can be modelled as if it was emanating from periodic sources

$$S_s(x,y) = \sum_{n=-\infty}^{\infty} Q_n^s \delta(x - nl)\,\delta(y) \tag{4}$$

The driving field is introduced in the Helmholtz equation for the scattered field

$$\Delta p_s + k^2 p_s = S_s(x,y) = \sum_{n=-\infty}^{\infty} Q_n^s \delta(x - nl)\,\delta(y) \tag{5}$$

In order to solve for the pressure dependence in the y-direction, $p(y)$, the periodicity of the sources along the x-axis is handled by a Fourier transform with respect to x. Hence

$$\begin{array}{c} \tilde{p}_{in} = 2\pi p_0 \delta(\alpha - k_x)\, e^{ik_y y} \;\; \partial \\ \left(-\alpha^2 + \dfrac{\partial^2}{\partial y^2} - k_z^2\right)\tilde{p}_s + k_0^2 \tilde{p}_s = \displaystyle\sum_{n=-\infty}^{+\infty} Q_n^s e^{i\alpha n l}\delta(y) \end{array} \tag{6}$$

3 - OBTAINING A CLOSED FORMULATION

By introducing the Poisson summation formula

$$2\pi a \sum_{k=-\infty}^{\infty} f\left(\frac{2\pi k}{a}\right) = \sum_{n=-\infty}^{\infty} F(na)\,,\;\; a > 0 \tag{7}$$

the infinite sum in equation (6) can be written as

$$\sum_{n=-\infty}^{\infty} Q_n^s e^{i\alpha n l} = \sum_{n=-\infty}^{\infty} Q_o^s e^{-inl(k_x - \alpha)} = Q_o^s 2\pi \sum_{n=-\infty}^{\infty} \delta(2\pi n - l(k_x - \alpha)) \tag{8}$$

and thus the Helmholtz equation becomes

$$\frac{\partial^2 \tilde{p}_s}{\partial y^2} + \underbrace{\left(k_0^2 - \alpha^2 - k_z^2\right)}_{-\beta^2}\tilde{p}_s = Q_o^s 2\pi \sum_{n=-\infty}^{\infty} \delta(2\pi n - l(k_x - \alpha))\,\delta(y) \tag{9}$$

The solutions of the two superposed one-dimensional equations (on the one hand the incoming wave together with the geometrically reflection and, on the other hand, the scattered field) can be calculated with e.g. a Laplace transform

$$\left\{ \begin{array}{ll} \text{Geometrically reflected wave :} & \dfrac{\partial^2 \tilde{p}_g}{\partial y^2} - \beta^2 \tilde{p}_g = 0 \\ \text{Scattered field :} & \dfrac{\partial^2 \tilde{p}_s}{\partial y^2} - \beta^2 \tilde{p}_s = \underbrace{2\pi Q_0^s \displaystyle\sum_{n=-\infty}^{\infty} \delta(2\pi n - l(k_x - \alpha))\,\delta(y)}_{D} \end{array} \right. \tag{10a}$$

$$\Rightarrow \left\{ \begin{array}{ll} \text{Geometricall reflected wave (homogeneous solution) :} & \tilde{p}_g = Be^{-\beta y} + Ce^{\beta y} \\ \text{Scattered field (particular solution) :} & \tilde{p}_s = -\dfrac{D}{\beta^2}\sinh(\beta y) \end{array} \right. \tag{10b}$$

The boundary condition at y=0 for the geometrical reflection, i. e. at $x \neq nL$, is:

$$\left[\frac{\partial(\tilde{p}_{in} + \tilde{p}_g)}{\partial y}\right]_{y=0} = -\beta(B - C) - 2i\pi k_y p_0 \delta(\alpha - k_x) = 0 \tag{11}$$

Under the condition that the geometrically reflected wave is moving outward, i. e. C=0, the pressure can be solved

$$B = \frac{-2i\pi k_y p_0 \delta\left(\alpha - k_x\right)}{\beta} \tag{12}$$

The scattered wave will move in the positive y-direction and therefore the sinh-term has to be modified in order not to increase to infinity.

$$\tilde{p}_s = \frac{D}{2\beta^2}\left(e^{\beta y} - e^{-\beta y}\right) \;\Rightarrow\; \tilde{p}_s = -\frac{De^{-\beta y}}{2\beta^2} \tag{13}$$

The expression for the pressure can now be inversely transformed

$$\begin{aligned} p\left(x,y\right) &= \frac{-k_y p_0}{\underbrace{\sqrt{k_0^2 - k_x^2 - k_z^2}}_{k_{ys}}} \exp i\left(k_x x - \underbrace{\sqrt{k_0^2 - k_x^2 - k_z^2}}_{k_{ys}}\, y\right) \\ &+ Q_0^s \sum_{n=-\infty}^{\infty} \frac{1}{l} \exp\left(\overbrace{\left(k_x - \frac{2\pi n}{l}\right)}^{k_{xn}} x - \overbrace{\sqrt{k_0^2 - \left(k_x - \frac{2\pi n}{l}\right)^2 - k_z^2}}^{k_{yn}}\, y\right) \\ &= -\frac{k_y}{k_{ys}} p_0 e^{i\left(k_x x + k_{ys} y\right)} + Q_0^s \sum_{n=-\infty}^{\infty} \frac{e^{i\left(k_{xn} x - k_{yn} y\right)}}{2k_{yn}^2} \end{aligned} \tag{14}$$

Here two aspects can be highlighted; on the one hand the fraction in the geometrical reflection will always be real. On the other hand, the terms in the infinite series will be real, and therefore radiate, as long as

$$-\frac{\left(\sqrt{k_0^2 - k_z^2} - k_x\right) l}{2\pi} < n < \frac{\left(\sqrt{k_0^2 - k_z^2} + k_x\right) l}{2\pi} \tag{15}$$

An equation similar to (9) but valid for the backside, $0<x<d$, has to be solved

$$\frac{\partial^2 \tilde{p}_b}{\partial y^2} + \underbrace{\left(k_0^2 - \alpha^2 - k_z^2\right)}_{-\beta^2} \tilde{p}_b = Q_0^b 2\pi \sum_{n=-\infty}^{\infty} \delta\left(2\pi n - l\left(k_x - \alpha\right)\right)\delta\left(y\right) \tag{16}$$

This equation is already solved (c.f. above) and has the solution

$$\tilde{p}_b = Be^{-\beta y} + Ce^{\beta y} - \frac{D}{\beta^2}\sinh\left(\beta y\right) \tag{17}$$

But here the boundary condition both at y=0 and at y=$-d$ are introduced

$$\left[\frac{\partial \tilde{p}_b}{\partial y}\right]_{y=0} = -\beta Be^{-\beta y} + \beta Ce^{\beta y} = \left(\beta\left(C - B\right)\right) = 0 \;\Rightarrow\; B = C \tag{18a}$$

$$\left[\frac{\partial \tilde{p}_b}{\partial y}\right]_{y=0} = B\left(e^{\beta y} - e^{-\beta y}\right) - \frac{D}{\beta}\sinh\left(\beta y\right) = 0 \;\Rightarrow\; \begin{cases} \beta d = 2\pi m, \;\; m = 0, \pm 1, \pm 2 \\ B = \dfrac{D}{\beta^2} \end{cases} \tag{18b}$$

and thus

$$\tilde{p}_b = \frac{D}{\beta^2}\left(2\cosh\left(\beta y\right) - \sinh\left(\beta y\right)\right) \tag{19}$$

which determines the natural frequencies for the geometrical reflection inside the cavity. However, the only present components at the backside field are emanating from the holes, i. e. the periodic velocity sources. What is interesting is the impedance, i.e. boundary condition, the velocity source will meet at the surface of the backside of the perforated plate. This will determine the magnitude of the velocity sources and thus the magnitude of the scattered field on the outside. The backing pressure can now be transformed back as for the scattered field

$$\tilde{p}_b = Q_o^b \sum_{n=-\infty}^{\infty} \frac{\left(2\cosh\left(k_{yn}y\right) - \sinh\left(k_{yn}y\right)\right) e^{ik_{xn}x}}{k_{yn}^2} \tag{20}$$

The volume velocity sources, Q_n, at $x{=}nL$, are governed by the relationship between the outer and the backing fields which can be represented as a two-port of a duct (the perforation) with the length l_h. The incorporated field quantities are pressure and volume velocity of the outer sound field on one side and of the backing sound field (indicated by the subscript b) on the other side

$$\begin{bmatrix} Q_s \\ Q_b \end{bmatrix} = \frac{iS}{(\rho c + RS)} \begin{bmatrix} \coth\left(\beta L\right) & \dfrac{-1}{\sinh\left(\beta L\right)} \\ \dfrac{-1}{\sinh\left(\beta L\right)} & \coth\left(\beta L\right) \end{bmatrix} \begin{bmatrix} p\,|_{y=0} \\ p_b\,|_{y=0} \end{bmatrix} \tag{21}$$

These equations can then be inserted into the equations for the pressure fields in order to obtain a single expression for the system.

4 - CONCLUSION

By solving the transformed sound field at the back and front side simultaneous, a formulation for the total system is obtained. Furthermore, the use of Poisson summation formula often makes it possible to reduce the sums to closed formulations. The technique is here demonstrated on the case of periodic slits, i. e. on a general two-dimensional problem.

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