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REGULARISATION METHODS FOR ACOUSTIC SOURCE RECONSTRUCTION

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ABSTRACT
This paper deals with the inverse source problem in acoustics. It is assumed that a number of acoustic sources are located at known spatial positions and the acoustic output is measured at a number of spatial positions in radiated sound field. An important inverse problem in the field of acoustics is that of reconstructing the strengths of a number of sources given a model of transmission paths from the sources to a number of receivers at which measurements are made. The accuracy of reconstruction of the source strength is dependent on the condition number of the matrix of transfer functions to be inverted. Therefore, some regularisation algorithms are often used to produce reasonable solutions to discrete ill-posed problems. But without prior knowledge of either the acoustic source field or the contaminating measurement noise, it is difficult to determine the optimal regularisation parameter. To improve the accuracy of reconstruction of acoustic sources by inverse techniques, this paper presents and compares the performance of a number of methods for choosing the optimal regularisation parameters by using a simple computer simulation model. In particular, we compare the use of generalised cross-validation and the L-curve method for a number of source-sensor geometrical arrangements.

1 - INTRODUCTION
An important inverse problem in the field of acoustics is that of reconstructing the strengths of a number of sources given a model of transmission paths from the sources to a number of sensors at which measurements are made. Here, the accuracy of reconstruction of the source strength is dependent on the conditioning of the matrix of transfer function to be inverted. However, in spite of an optimal geometrical arrangement of the sensors and sources, the transfer function matrix to be inverted may be ill-conditioned. This ill-conditioning will often result in an ill-posed problem. In such cases, by using only the simple least squares method, we cannot ensure a fine resolution of reconstruction of the acoustic source strength distribution. Therefore, some regularisation algorithms are often used to produce reasonable solutions to discrete ill-posed problems. But without prior knowledge of either the acoustic sources or the contaminating measurement noise, it is difficult to determine the proper regularisation parameter. Therefore, in order to improve the accuracy of reconstruction of acoustic sources, this paper will illustrate the performance of some regularisation methods for choosing the proper regularisation parameters through some results of a computer simulation of a particular geometry.

2 - CONDITIONING OF THE ACOUSTIC TRANSFER FUNCTION MATRIX
Sufficient guidelines to enable good resolution in reconstructing acoustic source strengths cannot be provided by using the condition number only [1]. Often, there is no approximate solution in the inverse process as a result to contamination noise, even though the transfer matrix is well-conditioned (i.e., the matrix has a small condition number). Hence, we need another constraint to be able to eliminate the effects of perturbations of the complex pressures such as, contamination of various kinds of errors (i.e., measurement errors and approximation errors), on the resolution of the reconstruction. By using the

least squares method based on the singular value decomposition, the reconstructed source strength vector $\mathbf{q}$ is given by

$$\mathbf{q} = \mathbf{G}^{-1}\mathbf{p} = \mathbf{V}\Sigma^{-\mathbf{1}}\mathbf{U}^H\mathbf{p} = \sum_{i=\mathbf{1}}^{N} \frac{\mathbf{u}_i^H\mathbf{p}}{\sigma_i}\mathbf{v}_i \tag{1}$$

where $\mathbf{p}$ represents the vector of complex acoustic pressure of the far field and the acoustic transfer function matrix $\mathbf{G}(= \mathbf{U\Sigma V^H})$ is assumed here to be a square matrix and the matrices $\mathbf{U}$ and $\mathbf{V}$ contain the left and right singular vector of $\mathbf{G}$. The superscript H denotes Hermitian transpose. In this equation, if the magnitude of the modulus of $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}$ is much greater than the associated singular value, $\mathbf{q}$ will be dominated by the terms in the sum corresponding to the smallest singular value. Moreover, the magnitude of the oscillatory $\mathbf{q}$ will be amplified by the right singular vector $\mathbf{v}_i$ (i.e., this reconstruction will suffer from the large perturbations caused by small perturbations of $\mathbf{p}$). Therefore, it is very important that, in order to reconstruct of the source strength distribution with accuracy, the effects of the contaminating noise on the role of the modulus of $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}$ in equation (1) must be understood and analysed. Fortunately, related to these effects, the discrete Picard condition [2] states that the magnitude variation of the modulus of $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}$, where $\mathbf{p}$ is the true complex pressure without contamination, must decay to zero faster than σ_i. More details of various applications of this condition can be found in reference [3]. In order to understand the relationship between the Picard condition and the conditioning of the acoustic transfer matrix, we have carried out some numerical simulations with the particular two-dimensional geometry shown in Fig. 1, where only one point cylindrically radiating monopole source located at the centre of the11 source line array is assumed to have unit strength. It is assumed that 10% and 20% measurement noise is added respectively to the exact complex pressure data at 11 measurement positions Here, it is assumed that the difference between the exact pressure $\mathbf{p}$ and the measured pressure $\hat{\mathbf{p}}$ including all types of errors is expressed as the vector of complex errors $\mathbf{e}$ given by $\mathbf{e} = \hat{\mathbf{p}} - \mathbf{p}$.

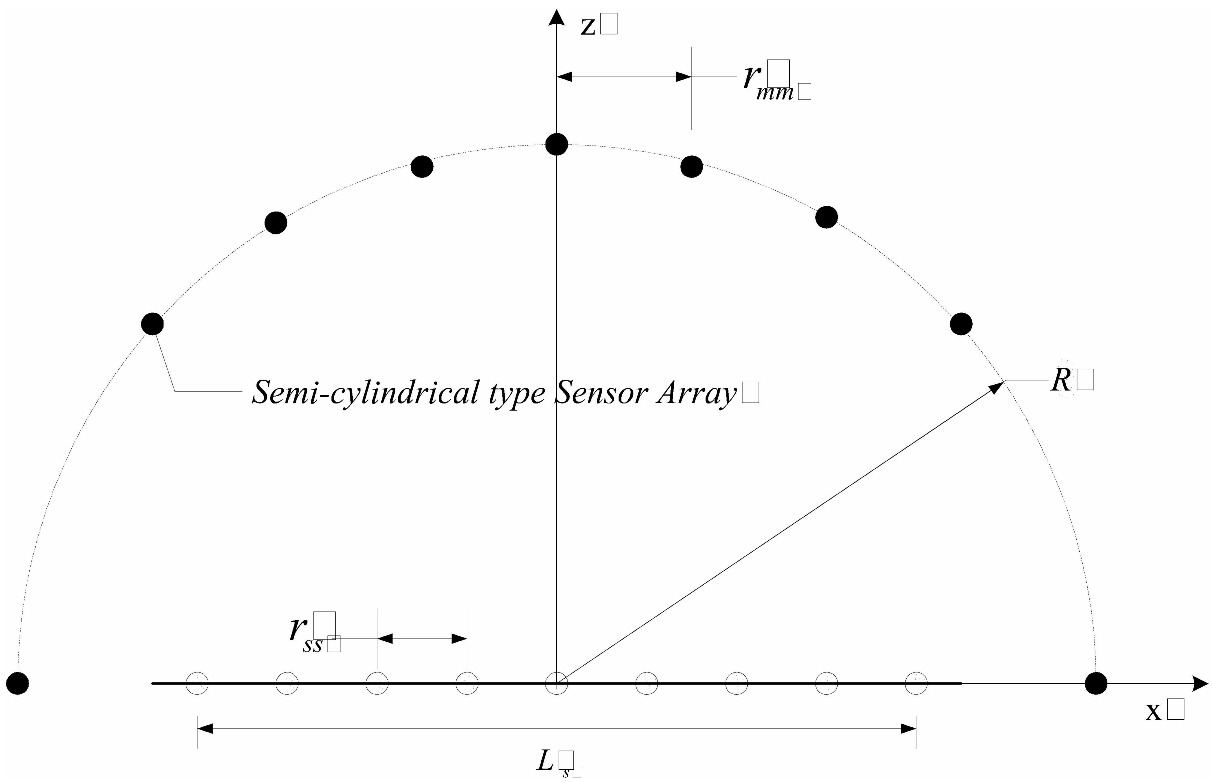

Figure 1: A particular geometrical arrangement of sensors and sources.

Errors are assumed to be spatially uncorrelated and were produced by 500 random trials with 10% and 20% of the magnitude of the true complex pressure. Fig. 3 shows magnitude variations of the modulus of $\mathbf{u}_i^{\mathrm{H}}\hat{\mathbf{p}}$ and the results of the reconstruction by the simple least squares method with respect to different levels of the contamination noise, when R=2L and the non-dimensional frequency $kr_{ss} = \pi/2$. Interestingly, in contrast to the magnitude variation of the modulus of $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}$ in Fig. 2, as the level of contamination noise increases, the value of the modulus of $\mathbf{u}_i^{\mathrm{H}}\hat{\mathbf{p}}$ increases rapidly increases in the region of the small singular values (i.e., the modulus of $\mathbf{u}_i^{\mathrm{H}}\hat{\mathbf{p}}$ does not satisfy the discrete Picard condition) and the resolution of the reconstruction becomes worse. This result is caused by the fact that the resolution of the reconstruction will be completely dependent on the value of the modulus of $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}/\sigma_i$ and the linear dependence between the estimated source strength $\mathbf{q}$ and the right singular vector $\mathbf{v}_i$ is destroyed by the very small singular values. Consequently, the dominant effect of noise on the variation of the modulus of $\mathbf{u}_i^{\mathrm{H}}\hat{\mathbf{p}}$ results in very poor reconstruction of the acoustic source distribution.

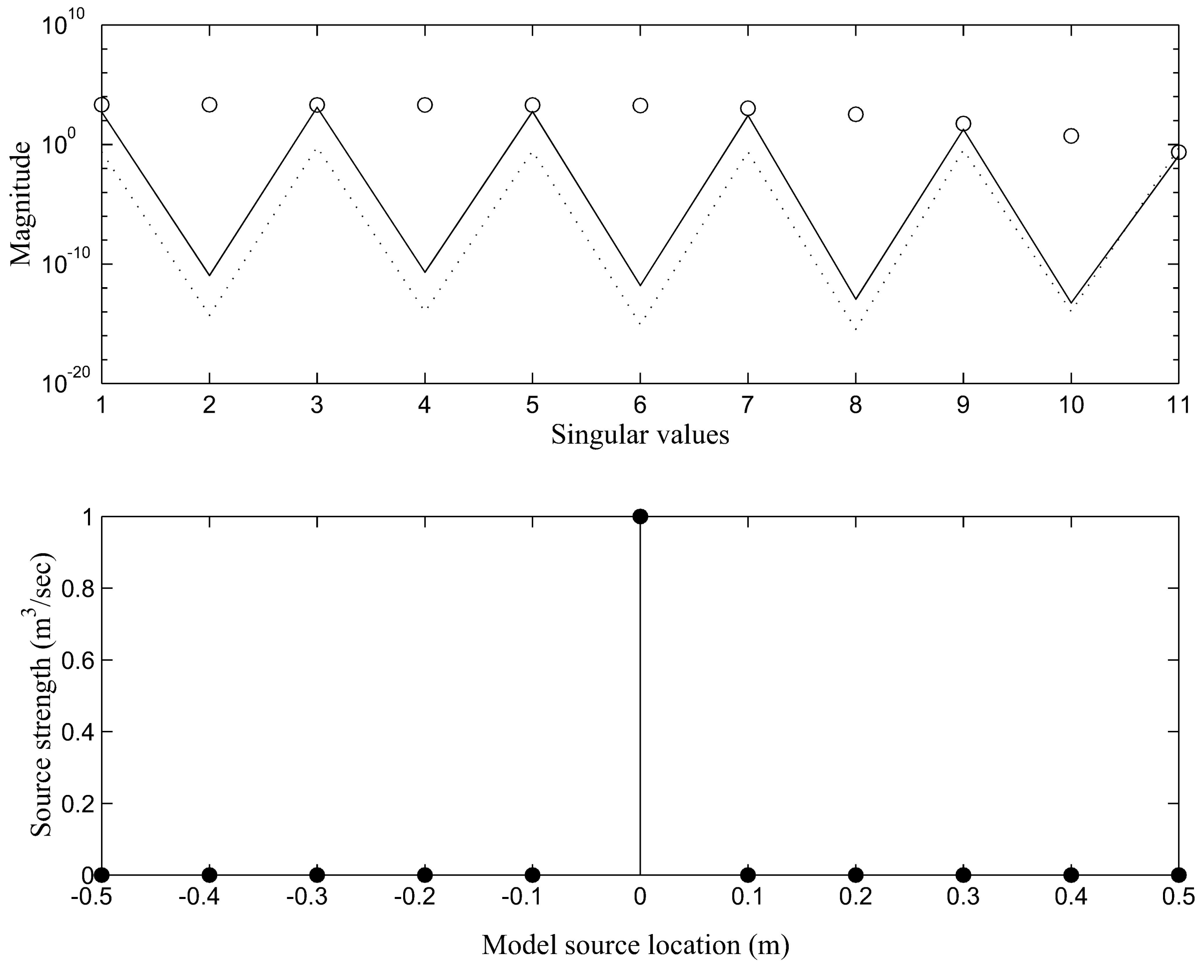

Figure 2: The magnitude variation of σ_i (circle), $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}$ (solid line) and $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}/\sigma_i$ (dotted line) of the true complex pressure and the reconstruction result produced by the simple least squares method.

3 - REGULARISATION METHODS FOR ACOUSTIC SOURCE RECONSTRUCTION

As shown above, it is necessary to incorporate further information about the reconstruction in order to improve the accuracy of reconstruction. If the transfer matrix is assumed to be a square matrix, the Tikhonov regularised reconstruction of the source strength vector $\mathbf{q}_0$ can be expressed by

$$\mathbf{q_0} = \mathbf{V}\Sigma_R^{-1}\mathbf{U}^H\hat{\mathbf{p}} = \sum_{i=1}^{N}\left(\frac{\sigma_i^2}{\sigma_i^2+\beta^2}\right)\frac{\mathbf{u}_i^H\hat{\mathbf{p}}}{\sigma_i}\mathbf{v}_i \tag{2}$$

where β denotes the chosen regularisation parameter. If β is determined properly, we can prevent the inversion of very small singular values. Therefore, the efficiency of the Tikhonov regularization method [4] depends on the proper choice of the regularization parameter β that produces a fair balance between the perturbation error and the regularisation error. However, for successful reconstruction, this regularisation method must have knowledge or a good estimation of error. In practical applications, it is very difficult to obtain prior knowledge or detailed estimation of the error. For this reason, we will introduce two techniques, Generalised Cross Validation [5] and the L-curve method [6], for determining the appropriate

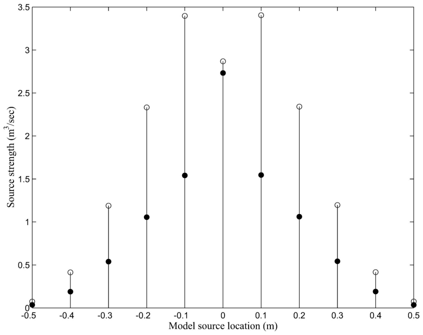

Figure 3(a): The reconstruction result produced by the simple least squares method, when adding 10% (filled circle) and 20% (blank circle) measurement noise respectively.

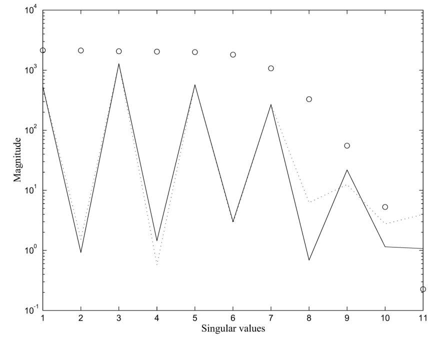

Figure 3(b): The magnitude variation of σ_i (circle), $\mathbf{u}_i^{\mathrm{H}}\hat{\mathbf{p}}$ (solid line: 10% contamination and dotted line: 20% contamination).

regularisation parameters. These techniques do not require prior knowledge of the source strength distribution or of the contamination noise. According to the full detail of the derivation in references [5], [7], the proper regularization parameter is determined by minimising the GCV function which can be defined by

$$GCV\,(\beta) = \frac{(1/M)\,\|\{\mathbf{I}{-}\mathbf{B}(\beta)\;\;\hat{\mathbf{p}}\}\|^2}{[(1/M)\,\mathrm{trace}\,\{\;\mathbf{I}{-}\mathbf{B}(\beta)\}]^2} \tag{3}$$

where $\mathbf{B}\,(\beta)$, the so called influence matrix, is expressed by $\mathbf{B}\,(\beta) = \mathbf{G}\left(\mathbf{G^H G} + \beta^2\mathbf{I}\right)^{-1}\mathbf{G^H}$. Another convenient tool for determining the appropriate regularisation parameters of discrete ill-posed problems without prior information is the so-called L-curve method [2]. This is a graphical tool with a plot of the regularised solution against its residual for all valid regularisation parameters. The appropriate regularisation parameter β corresponds to the maximum curvature of the L-shaped appearance and is defined by [6]

$$L\,(\beta) = \frac{\tilde{\rho}'\tilde{\eta}'' - \tilde{\rho}''\tilde{\eta}'}{\left((\tilde{\rho}')^2 + (\tilde{\eta}')^2\right)^{3/2}} \tag{4}$$

where $\eta = \|\mathbf{q}_R\|^2,\ \rho = \|\mathbf{G}\mathbf{q}_R - \mathbf{p}\|^2$. Also $\tilde{\eta} = \log\eta,\ \tilde{\rho} = \log\rho$ and the prime denotes differentiation with respect to the regularisation parameter β. We have applied these regularisation methods (which includes the determination of the proper regularisation parameter) to the same geometry of Fig. 1. Fig. 4(a) shows that good results for the reconstruction by using Tikhonov regularisation with the GCV technique with two different levels of contamination noise. These can be compared with Fig. 3(a) by the simple least squares method. In the region of the small singular values in Fig. 4(b), the magnitude variation of the modulus of $\mathbf{u}_i^{\mathrm{H}}\mathbf{p}/\sigma_i$ has been properly suppressed by the chosen regularisation parameter compared to that from the simple least squares method in Fig. 3(b). Here, the value of β for the 10% contamination is smaller than that for 20% contamination; in other words, it is necessary to increase the regularisation for a successful reconstruction with high levels of contaminating measurement noise. It can be also shown in Fig. 5 that, in the case of application of Tikhonov regularisation with the L-curve method for the same conditions, good reconstruction is presented as a result of choosing the appropriate regularization parameters.

4 - CONCLUSIONS

We have introduced the discrete Picard condition, which defines the effects of noise contamination on the accuracy and the resolution of reconstruction of source strength. We have applied some well-known regularisation methods including the determination of the proper regularisation parameters without prior information to the simple geometry. By the application of the discrete Picard condition, it can be seen easily how chosen regularisation parameters work in the reconstruction of source strength. In addition, it may become an important criterion for good resolution of reconstruction. Through numerical simulations, we have illustrated that the application of regularisation methods introduced in this paper

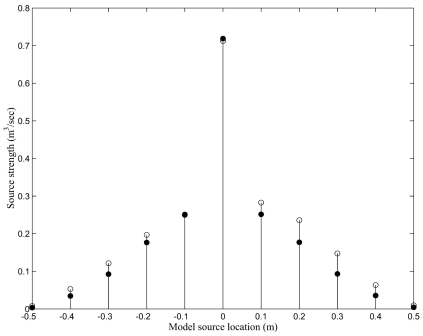

Figure 4(a): The reconstruction result produced by Tikhonov regularisation method with GCV, when adding 10% (filled circle) and 20% (blank circle) measurement noise respectively.

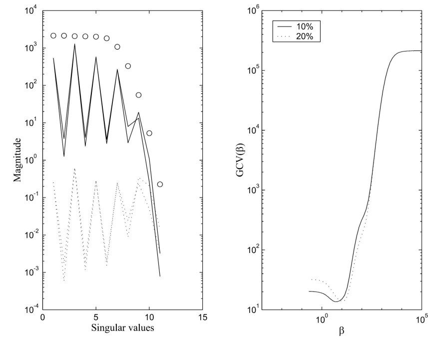

Figure 4(b): The magnitude variation of σ_i (circle), $\mathbf{u}_i^{\mathrm{H}}\hat{\mathbf{p}}$ and the variation of the GCV function with (solid line: 10% contamination and dotted line: 20% contamination).

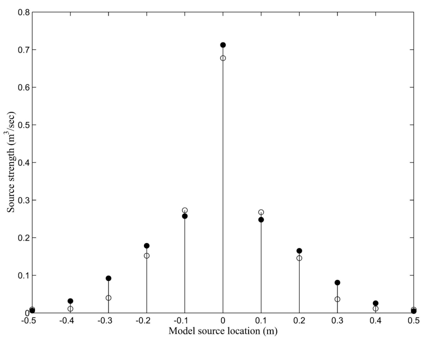

Figure 5(a): The reconstruction result produced by Tikhonov regularisation method with the L-curve method, when adding 10% (filled circle) and 20% (blank circle) measurement noise respectively.

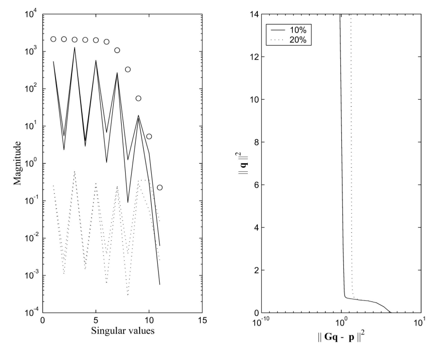

Figure 5(b): The magnitude variation of σ_i (circle), $\mathbf{u}_i^{\mathrm{H}}\hat{\mathbf{p}}$ and the L-shape curve (solid line: 10% contamination and dotted line: 20% contamination).

to an ill-posed acoustical inverse problem can provide considerable improvement in the accuracy and the resolution of the reconstruction.

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