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REGULARIZATION AND NEARFIELD ACOUSTICAL HOLOGRAPHY

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ABSTRACT
The reconstruction of the pressure and normal surface velocity provided by nearfield acoustical holography (NAH) from pressure measurements made near a vibrating structure is a linear, ill-posed inverse problem. Regularization provides a technique of overcoming the ill-posedness to generate a solution to the linear problem in an automated way. Our objectives are (1) to develop an approach which formulates exterior and interior NAH problems in an identical way so that a single regularization theory may be applied to both these problems and (2) using model data, representative of a typical NAH measurement, we compare the errors and filter factors associated with four regularization schemes; Tikhonov, truncated singular value, conjugate gradient and Landweber iteration given various levels of signal to noise in the pressure data. We use the Morozov discrepancy principle for parameter estimation.

1 - INTRODUCTION
The theory of regularization for linear problems is a heavily researched area and many books [1] are available on the subject. However it is clear that there is no holy grail as too the best approach, and the success of any particular approach depends on the nature and the physics of the problem which is being solved. NAH is distinguished by the existence of evanescent waves which decay exponentially from the surface of the vibrator. This decay is the root of the ill-posed nature of the inverse problem which turns these exponential decays into exponential amplifications reeking havoc due to noise contained in the measured data.
With regard to regularization approaches one thing is clear, the singular value decomposition (SVD) of the system matrix plays a crucial role. We provide in Section 2 a formulation of the exterior, separable geometry NAH problem which casts it as a singular value decomposition so that it can be treated in the same way as the conformal exterior NAH and interior NAH problems, both which have been solved using the SVD. This formulation brings all of the NAH problems under the same umbrella so that they can be treated equally with a single regularization approach. In Section 3 we discuss four regularization schemes and in Sec. 4 apply them to some numerical data generated to be representative of the evanescent wave nature of the NAH problem. The reconstruction errors and reconstruction filters are compared.

2 - SEPARABLE GEOMETRY NAH AND THE SVD
Let $\dot{w}_d$ be the discretized normal velocity of a separable (planar, cylindrical or spherical) vibrating surface *S'* and p_d be the discretized hologram (measured) pressure on the surface S located at a constant distance d from *S'*. The following relationship exists between them [2]:

$$p_d = F_d^{-1} G_{Nd} F_d \dot{w}_d \tag{1}$$

where F_d represents a two-dimensional discrete Fourier transform. We call $F_d \dot{w}_d$ and $F_d p_d$ the k-space representations of the velocity and pressure, respectively. Here p_d is a column vector $\left(p_d \in C^M\right)$, $\dot{w}_d$ $\left(\dot{w}_d \in C^N\right)$ a column vector of normal velocity and G_{Nd} $\left(G_{Nd} \in C^{M\times N}\right)$ is the diagonal Neumann matrix.
Since the operations given by F_d^{-1} and F_d are just matrix multiplications we can define a spatial transfer function $H^{M\times N}$ which relates directly the pressure vector to the velocity vector:

$$p_d = H\dot{w}_d \tag{2}$$

where H embodies the DFT operations in Eq. (1); $H \equiv F_d^{-1} G_{Nd} F_{d.}$.
Now we can consider the singular value decomposition (SVD) of the transfer function H in Eq. (2), given by

$$H = U \Sigma V^H \tag{3}$$

where U and V are left and right unitary (orthonormal) matrices, respectively, and Σ is a diagonal matrix of complex singular values. Comparison of Eq. (3) with Eq. (1) reveals that the SVD and Fourier transform decomposition are one in the same thing, as long as we allow the singular values to be complex numbers. Thus $V^H = F_d, U = F_d^{-1}$ and $\Sigma = G_{Nd}$. For example, the singular values for the planar geometry are just

$$\sigma_{ij} = \frac{\rho c k}{\sqrt{k^2 - k_{xi} - k_{yj}}} e^{i\sqrt{k^2 - k_{xi} - k_{yj}}(z - z')} \qquad \text{with} \qquad \Sigma \equiv diag\left[\sigma_{11}, \sigma_{12} \ldots, \sigma_{ij}, \ldots\right]$$

given the discretization of $(k_x, k_y) \rightarrow (k_{xi}, k_{yi})$ where diag is a diagonal matrix and (i,j) spans all the k-space components used in the discretization. When $k_{cij} \equiv \sqrt{k_{xi} + k_{yj}} > k$ (k_{cij} lies outside the radiation circle) the magnitudes of the singular values decay exponentially.

3 - REGULARIZATION

The literature for regularization of linear equations, like Eq. (2), is vast and many books have been written on the subject. Unfortunately, there appears to be no holly grail with respect to the best technique, since the characteristics of the inversion depend very much on the physics of the problem. Applications to NAH problems are just beginning to appear in the literature. We concentrate on regularization techniques based on a knowledge of the noise variance, and present a rather simple method of determining the noise from the data. We discuss four regularization techniques; the standard Tikhonov method, Landweber iteration, the conjugate gradient approach, and a modified truncated SVD approach. In all these approaches one must also apply a stopping rule or parameter selection method. We use the most popular one, the discrepancy principle of Morozov.
To formulate the effects of noise in the hologram data p_d, assume that spatially uncorrelated random noise ε with zero mean and variance σ is present in the measurement. We use a superscript δ to indicate a quantity with noise. Thus p_d is the pressure data with noise. We rewrite Eq. (2) including noise as (E is expectation)

$$p_d^\delta = H\dot{w}_d^\delta,\ p_d^\delta = p_d + \varepsilon \qquad \text{and} \qquad E\left\|p_d - p_d^\delta\right\| = \sigma\sqrt{M} \tag{4}$$

3.1 - Tikhonov regularization

Tikhonov regularization minimizes $\left\|H\dot{w}_d^\delta - p_d^\delta\right\|^2 + \alpha\left\|\dot{w}_d^\delta\right\|^2$ ($\|\cdot\|$ represents the $L2$ norm). We use the discrepancy principal to determine the regularization parameter α. The solution using Eq. (3) is well known and is given by $\dot{w}_d^\delta = R_\alpha p_d^\delta$ where R_α is called the regularized inverse of H and

$$R_\alpha = V F^\alpha diag\left(\frac{1}{\sigma_1}, ..., \frac{1}{\sigma_N}\right) U^H, \qquad F^\alpha = diag\left(\frac{|\sigma_1|^2}{\alpha + |\sigma_1|^2}, ..., \frac{|\sigma_N|^2}{\alpha + |\sigma_N|^2}\right) \tag{5}$$

F^α is the k-space filter factor. The predicted value of the hologram pressure $p_d^{\alpha\delta}$ (now depending on α and the reconstructed velocity) is

$$p_d^{\alpha\delta} \equiv H\dot{w}_d^{\alpha\delta} = H R_\alpha p_d^\delta = U F^\alpha U^H p_d^\delta \tag{6}$$

The Morozov discrepancy equation is

$$\left\|H\dot{w}_d^{\alpha\delta} - p_d^\delta\right\| = \left\|p_d^{\alpha\delta} - p_d^\delta\right\| = \delta,\ \delta = \sqrt{M\sigma} \tag{7}$$

that is, the smoothed pressure should differ from the measured pressure by the noise variance. This mimics the condition shown in Eq. (4) for the noise variance. Using the SVD we can write this equation in k-space as

$$\left\| F_h \bar{p}_d^\delta \right\| = \delta, \qquad \bar{p}_d^\delta \equiv U^H p_d^\delta, \qquad F_h = diag\left(\frac{\alpha}{\alpha + |\sigma_1|^2}, ..., \frac{\alpha}{\alpha + |\sigma_N|^2} \right) \tag{8}$$

We recognize $\bar{p}_d^\delta$ as the k-space representation of the measured pressure. F_h^α is a high pass filter admitting only small singular values. We use this version of the discrepancy principle for all four regularization techniques presented here, with separate expressions for the filter factor and with α representing the corresponding regularization parameter of the particular method. The discrepancy equation is robust computationally; the left hand side represents a monotonic function in α.

3.2 - Landweber iteration

This popular technique [4] provides the solution (integer regularization parameter m)

$$\dot{w}^m = R_m p = V diag\left(\frac{1 - \left(1 - \beta |\sigma_1|^2\right)^m}{\sigma_1}, ..., \frac{1 - \left(1 - \beta |\sigma_N|^2\right)^m}{\sigma_N} \right) U^H p \tag{9}$$

from which, in analogy to Eq. (5), we can derive the filter factor F^m:

$$R_m = V F^m diag\left(\frac{1}{\sigma_1}, ..., \frac{1}{\sigma_N} \right) U^H, \; F^m = diag\left(1 - \left(1 - \beta |\sigma_1|^2\right)^m, ..., 1 - \left(1 - \beta |\sigma_N|^2\right)^m\right) \tag{10}$$

In order for convergence to occur we must choose $\beta < 1/ |\sigma_{\max}|^2$. The high pass filter for the discrepancy principle is just

$$F_h^m \equiv I - F^m = diag\left(\left(1 - \beta |\sigma_1|^2\right)^m, ..., \left(1 - \beta |\sigma_N|^2\right)^m\right) \tag{11}$$

3.3 - Conjugate gradient approach

The method used here is referred to as CGLS [1]. The algorithm consists of five statements, not reproduced here, which are easily programmed. These statements were translated into k-space using Eq. (3). Again the Morozov discrepancy principle was applied to find a stopping point in the interactions. Since no explicit form of the filter factor F is available it is determined from Eq. (6), where α represents the iteration number, by comparison of the measured pressure to the filtered hologram pressure resulting from the reconstructed velocity. F remains diagonal in the CG approach.

3.4 - Determination of variance of the hologram noise

As pointed out by Hansen [1] the expected value of the Fourier coefficients of the noise is given by

$$E\left(\left|U_i^H \varepsilon\right|\right) = \sigma, \; i = 1, ..., N \tag{12}$$

so that the Fourier coefficients of the measured pressure will level off at

$$E\left(\left|U_q^H p_d^\delta\right|\right) \approx \sigma \text{ for large } q \tag{13}$$

since the Fourier coefficients are dominated by $\left|U_q^H \varepsilon\right|$ for large q. Thus to determine the variance σ we choose a set of Fourier coefficients for a given set of values of q at the extremes of the discretized k-space spectrum, and take a mean-square average of the set. This technique is completely general and will work in any geometry, for interior or exterior NAH problems. It was initially used successfully in the interior of an aircraft fuselage [3]. The noise added in the simulations below was always successfully reproduced by this technique.

4 - COMPARISON OF RESULTS

The numerical model consided of four linear arrays of 36 point sources located slightly within/below the reconstruction plane. Each array was phased to be subsonic with phase speeds of -300, 450, -600, 750 m/s in water with c=1481 m/s. Thus adjacent point sources are in acoustic short circuit and create radiating fields which tend to decay evanescently. The normal velocity was computed exactly on the reconstruction plane. The hologram was placed 1.3 cm from the reconstruction plane. Random gaussian noise was added to the hologram with predetermined variance to simulate measurements with 40, 25 and 15dB signal-to-noise ratios at 2kHz. Using the pseudo-inverse of H the reconstruction error $= \left\| \dot{w}_{ex} - \dot{w}_d^{\alpha\delta} \right\| / \left\| \dot{w}_{ex} \right\|$ was 2.2% for the no noise case, which requires no regularization.

The results of the comparison of the four regularization procedures is shown in figs. 1, 2 and 3 which present results for SNRs of 40, 25 and 15dB respectively. In these figures the horizontal axis is the radius of the k-space circle, $k_c = \sqrt{k_x^2 + k_y^2}$ where k_x and k_y are the wavenumbers in the two orthogonal coordinate directions. An approximate relation exists with the Tikhonov factor α: $k_c \propto |\log_{10}\alpha|$. In fact, one can view k_c as proportional to the regularization parameter for each of the approaches. The vertical axis is the percent error between the reconstructed normal velocity and the known solution.

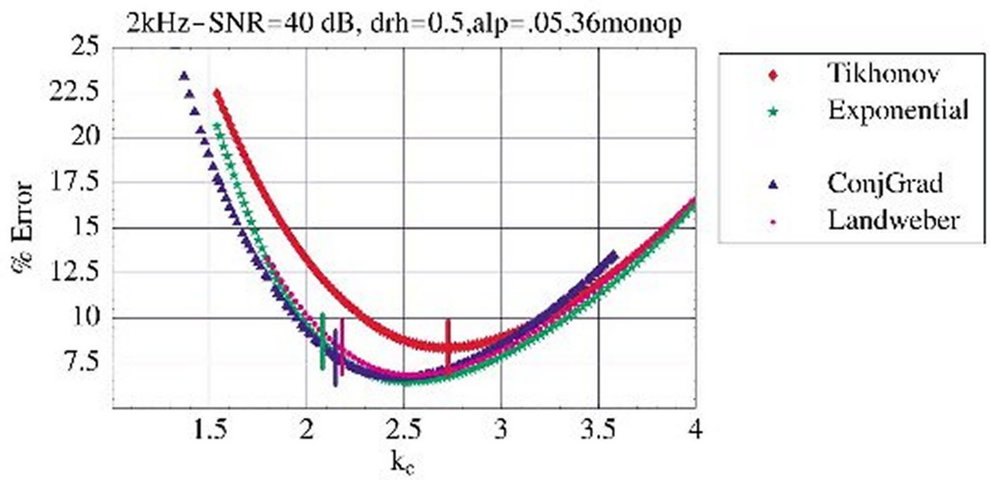

Figure 1: Comparison of percent errors versus cutoff wavenumber $\left(k_c = \sqrt{k_x^2 + k_y^2}\right)$ for four regularization techniques given SNR=40 dB; the short vertical lines indicate the cutoff determined by the Morozov discrepancy principle.

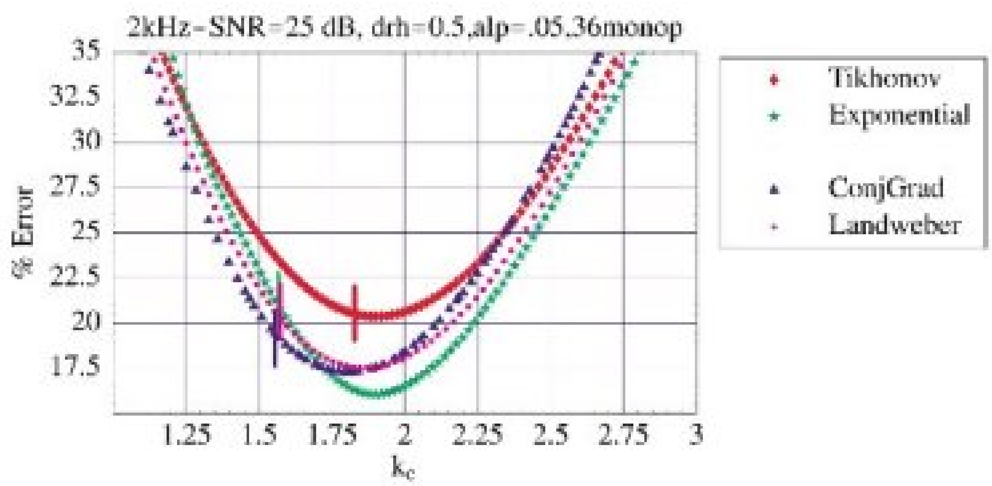

Figure 2: Same as fig. 1 except that the SNR = 25dB.

The baseline for comparison is the modified truncated SVD solution which corresponds to the curves labeled 'Exponential', and uses a very steep window with an exponential taper and break point at k_c. We believe that this baseline solution represents the best which can be achieved. Two conclusions are evident from these figures. (1) Tikhonov regularization does not do as well as Landweber (LW) and CG, (2) the discrepancy principle used for the stopping the search always oversmooths (smaller values of k_c) the solution, missing the actual minimum. The latter is a well known fact. However, the discrepancy principle appears to locate closer to the minimum for the Tikhonov solution, so that the resulting error is close to the stop point error for the other approaches. We also conclude when the SNR is high, all the approaches seem to do quite well, and as the SNR drops both the CG and LW reach about the same minimum error.

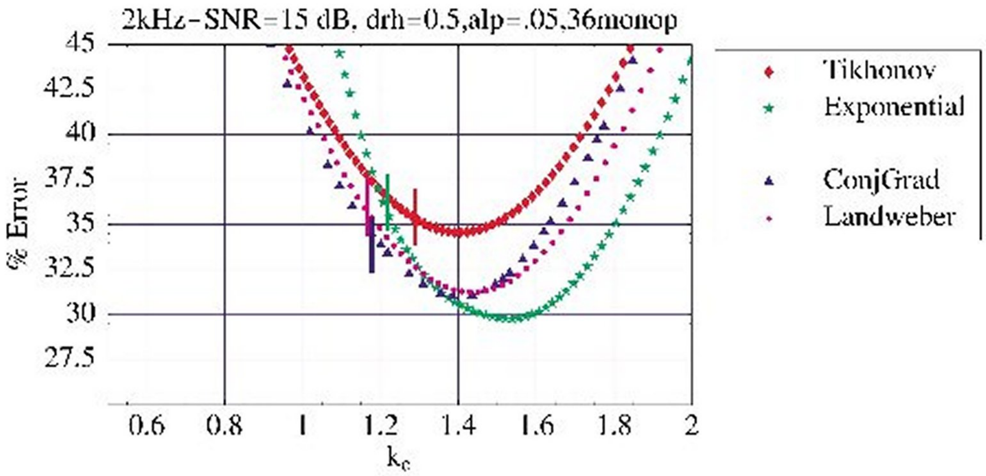

Figure 3: Same as fig. 1 except that the SNR = 15dB.

Comparison of filter factors

It is very instructive to look at the tapers involved with the various k-space filters which arise from the four procedures. The next figure compares the four filters plotting the filter factor F against k_c for a SNR of 25dB. There is a clear correlation between the steepness of the filter and the errors shown in fig. 2; it is apparent that the Tikhonov solution is the poorest of the four due to the broad nature of the filter taper. The CG and LW filters are nearly the same except for the oscillation of the CG filter to levels above F=1. The overshoot is a surprising and undesirable feature of the CG filter.

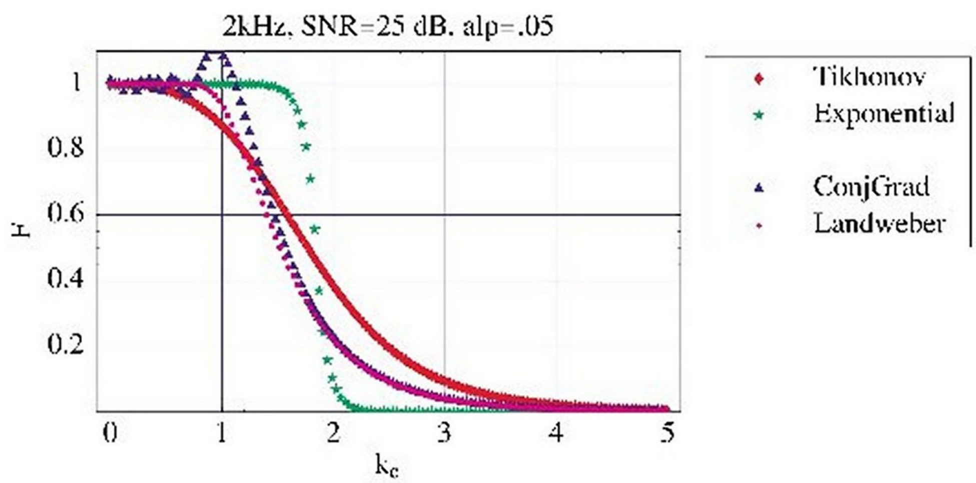

Figure 4: The filter shapes for the four cases with SNR=25dB; the break points of the filters are determined by the discrepancy principle, corresponding to the vertical lines in fig. 2; the baseline filter is the exponential one chosen with a steep break.

5 - CONCLUSIONS

All of the approaches coupled with the Morozov discrepancy principle are successful at regularization, and the small differences in errors may be of little concern to those whose applications do not warrant high precision. At the risk, then, of splitting hairs, it appears that the Landweber and the conjugate gradient approaches to regularization do equally well when compared with the baseline solution with mimina which are only a few percent higher than the best case. Thus we would recommend them for the general NAH problem. We are less happy with the stopping rule provided by the Morozov discrepancy principle which appears to miss the minima by several percent (except for Tikhonov). There exist in the literature refinements to the discrepancy principle which should be exercised. There are parameter

selection methods based on estimation of the reconstruction error which appear promising but were not investigated for this paper.
It would appear that the truncated SVD solution simulated by the exponential filter with a small amount of taper is the best solution. For this case, however, it appears that the stopping rule does worse that the other three cases, so a better stopping rule could make this the best method for regularization.

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