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CLOSED EQUATIONS FOR STATISTICAL MOMENTS OF A SOUND FIELD IN A TURBULENT, REFRACTIVE ATMOSPHERE NEAR AN IMPEDANCE GROUND AND THEIR SOLUTIONS

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ABSTRACT
A parabolic equation for a sound pressure field is now widely used for numerical simulations of sound propagation near an impedance ground in a turbulent, stratified atmosphere. For different realizations of the temperature and wind velocity fields, the parabolic equation is solved numerically. Then, the statistical moments of a sound field are calculated from the ensemble of sound pressure realizations. We employ a different approach for calculating these statistical moments that can be advantageous in many cases. Starting from a parabolic equation and using the Markov approximation, we derive closed equations for statistical moments (of arbitrary order) of a sound field propagating near impedance ground in a turbulent, stratified atmosphere. Then, the derived equations are solved analytically or numerically.

1 - INTRODUCTION
Studies of sound propagation over impedance ground in a turbulent, stratified atmosphere are important because they help to address many concerns of atmospheric acoustics such as source detection, ranging, classification, noise pollution near highways and airports, etc. These studies are rather involved and are usually performed by the following approach (e.g., [1,2]). A realization of random fields of temperature $T'\left(\vec{R}\right)$ and the x-component $v_x\left(\vec{R}\right)$ of wind velocity fluctuations is synthesized numerically. Here, $\vec{R}=(x,y,z)$, where x, y and z are the Cartesian coordinates with the x-axis in the direction from the source to receiver, z-axis directed upward, and the plane $z=0$ coinciding with the surface of the ground. These random fields are superimposed on mean vertical profiles of sound speed $c(z)$ and wind velocity vector $\vec{V}(z)$. Then, a realization of the sound field p due to a monochromatic sound source in such a turbulent, refractive atmosphere is calculated numerically by using the parabolic equation. This procedure is repeated for many realizations of $T'\left(\vec{R}\right)$ and $v_x\left(\vec{R}\right)$. Finally, the statistical moments of the sound field are calculated from realizations of p.
The main idea of the present paper is to develop a simpler approach for calculating the statistical moments of p in a turbulent, stratified atmosphere. In this approach, first we derive closed analytical equations for the statistical moments of p and only then do we solve these equations numerically or analytically. For electromagnetic wave propagation over an impedance boundary in a turbulent, refractive atmosphere, a similar approach was considered in references [3,4].

2 - PARABOLIC EQUATION
Let the source be located in the plane $x=0$. The sound pressure p due to the source satisfies boundary conditions and Eq. (6.1) from [5]. The right hand-side of this equation contains the source term. We

assume that for $x > 0$ solution of this equation can be approximated by solution of the following parabolic equation [5]:

$$[\partial/\partial x + M(\vec{r}) - (ik_0/2)\,\varepsilon(x,\vec{r})]\,p(x,\vec{r}) = 0. \tag{1}$$

Here, $\vec{r} = (y,z)$ are the transverse coordinates; $\varepsilon = -T'/T_0 - 2v_x/c_0$ is a linear combination of temperature and wind velocity fluctuations; k_0, T_0 and c_0 are reference values of the sound wavenumber, temperature, and sound speed; and the operator M is given by:

$$M = (2ik_0)^{-1}\left(k^2 + k_0^2 + \nabla_\perp^2\right) + \vec{V}_\perp \cdot \nabla_\perp/c_0, \tag{2}$$

where $k\left(\vec{R}\right) = k_0c_0/\left[c\left(\vec{R}\right) + V_x\left(\vec{R}\right)\right]$ is the sound wavenumber, $V_x\left(\vec{R}\right)$ and $\vec{V}_\perp\left(\vec{R}\right)$ are the components of the mean velocity vector $\vec{V}\left(\vec{R}\right)$ in the direction of the x-axis and perpendicular to it, and $\nabla_\perp = (\partial/\partial y, \partial/\partial z)$. Equation (1) should be supplemented by the initial condition $p(x=0,\vec{r}) = p_0(\vec{r})$ which is assumed to be known and the following boundary condition at $z = 0$:

$$(\partial/\partial z + ik_0\beta)\,p(x,y,z) = 0, \tag{3}$$

where β is the normalized admittance of the ground.
When deriving the statistical moments of p, we will assume that the random field $\varepsilon\left(\vec{R}\right)$ is δ-correlated along the x-axis. In this case, which is also known as the Markov approximation [6], the correlation function $B(x,\vec{r};x',\vec{r}') = \langle\varepsilon(x,\vec{r})\,\varepsilon(x',\vec{r}')\rangle$ of the random field ε is given by $B(x,\vec{r};x',\vec{r}') = \delta(x-x')\,b(x;\vec{r};\vec{r}')$. Here, δ is the delta function. Detailed consideration of the correlation functions B and b can be found in section 7.5 of Ref. [5].
Starting from parabolic equation (1) and using the Markov approximation, we have derived closed equations for the statistical moments of a sound field propagating near the impedance ground in a turbulent, stratified atmosphere. The subsequent sections present such closed equations and their analysis for the mean sound field $\langle p\rangle$, the transverse coherence function $\Gamma(x;\vec{r}_1;\vec{r}_2) = \langle p(x,\vec{r}_1)\,p^*(x,\vec{r}_2)\rangle$, and the statistical moment of order $n+m$:

$$\Gamma(x;\vec{r}_1,...,\vec{r}_n;\vec{r}_1',...,\vec{r}_m') = \langle p(x,\vec{r}_1)...p(x,\vec{r}_n)\,p^*(x,\vec{r}_1')...p^*(x,\vec{r}_m')\rangle\,. \tag{4}$$

3 - MEAN SOUND FIELD

A closed equation for the mean sound field is given by:

$$\left[\partial/\partial x - (i/2k_0)\left(k^2 + k_0^2 + \nabla_\perp^2\right) + \vec{V}_\perp \cdot \nabla_\perp/c_0 + \left(k_0^2/8\right) b(x;\vec{r};\vec{r})\right]\langle p(x,\vec{r})\rangle = 0. \tag{5}$$

This equation should be supplemented by the initial condition $\langle p(x=0,\vec{r})\rangle = p_0(\vec{r})$ and the boundary condition at $z = 0$, which follows directly from Eq. (3):

$$(\partial/\partial z + ik_0\beta)\,\langle p(x,y,z)\rangle = 0. \tag{6}$$

Equation (5) is valid for anisotropic, inhomogeneous turbulence as well as for isotropic, homogeneous turbulence. For the latter case, $b(x;\vec{r};\vec{r}') = b(\vec{r}-\vec{r}')$ so that in Eq. (5) $b(x;\vec{r};\vec{r}) = b(0)$. In this case, Eq. (5) has the following solution:

$$\langle p(x,\vec{r})\rangle = \exp\left(k_0^2 b(0)\,x/8\right) p^{(0)}(x,\vec{r})\,. \tag{7}$$

Here, $p^{(0)}(x,\vec{r})$ is the sound field that would be observed in the absence of random inhomogeneities. The field $p^{(0)}(x,\vec{r})$ satisfies Eq. (5) with $b = 0$ and the same boundary and initial conditions as for $\langle p(x,\vec{r})\rangle$. According to Eq. (7), the mean sound field attenuates exponentially. Note that Eq. (7) has the same form as a formula for the mean sound field in an unbounded turbulent atmosphere, see Eq. (7.59) from [5].

4 - COHERENCE FUNCTION

A closed equation for the coherence function Γ of a sound field is given by

$$\frac{\partial \Gamma\left(x; \vec{r}_1, \vec{r}_2\right)}{\partial x} - \frac{i}{2k_0}\left[\nabla^2_{\perp,1} - \nabla^2_{\perp,2} + k^2\left(x, \vec{r}_1\right) - k^2\left(x, \vec{r}_2\right) + \frac{2ik_0}{c_0}\left(\vec{V}_\perp\left(x, \vec{r}_1\right)\cdot\nabla_{\perp,1} + \vec{V}_\perp\left(x, \vec{r}_2\right)\cdot\nabla_{\perp,2}\right)\right]$$
$$\times\Gamma\left(x; \vec{r}_1, \vec{r}_2\right) + \frac{k_0^2}{8}\left[b\left(x; \vec{r}_1; \vec{r}_1\right) + b\left(x; \vec{r}_2; \vec{r}_2\right) - 2b\left(x; \vec{r}_1; \vec{r}_2\right)\right]\Gamma\left(x; \vec{r}_1, \vec{r}_2\right) = 0. \tag{8}$$

Here, $\nabla_{\perp,1} = (\partial/\partial y_1, \partial/\partial z_1)$ and $\nabla_{\perp,2} = (\partial/\partial y_2, \partial/\partial z_2)$. The coherence function Γ satisfies the following initial condition: $\Gamma\left(x = 0; \vec{r}_1; \vec{r}_2\right) = p_0\left(\vec{r}_1\right) p_0^*\left(\vec{r}_2\right)$. Also, Γ should satisfy the boundary condition

$$\left(\partial/\partial z_1 + ik_0\beta\right)\Gamma\left(x; \vec{r}_1; \vec{r}_2\right) = 0 \tag{9}$$

at $z_1 = 0$, and the boundary condition

$$\left(\partial/\partial z_2 - ik_0\beta^*\right)\Gamma\left(x; \vec{r}_1; \vec{r}_2\right) = 0 \tag{10}$$

at $z_2 = 0$. At present, we do not know whether Eq. (8) can be solved analytically. It could be possible to obtain approximate analytical solutions of this equation. We have, however, succeeded in devising numerical methods for solving this equation directly in a turbulent, stratified atmosphere, where k depends only on z. When the two-dimensional version of Eq. (8) is discretized in the vertical coordinate with mesh spacing Δz, one obtains

$$\begin{aligned}\frac{\partial \Gamma\left(x; z_m; z_n\right)}{\partial x} = \frac{i}{2k_0\Delta z^2}&\left[\Gamma\left(x; z_{m-1}; z_n\right) + \Gamma\left(x; z_{m+1}; z_n\right) - \Gamma\left(x; z_m; z_{n-1}\right) - \Gamma\left(x; z_m; z_{n+1}\right)\right] \\ &+ \left(a_m + a_n^* - d_{mn}\right)\Gamma\left(x; z_m; z_n\right),\end{aligned} \tag{11}$$

where $a_n = i\left[k^2\left(z_n\right) - k_0^2\right]/2k_0$ and $d_{mn} = b\left(z_m, z_m\right) + b\left(z_n, z_n\right) - 2b\left(z_m, z_n\right)$. The indices m and n range from 1 to N, where N is the number of vertical layers. Numerical implementation of the boundary conditions for G is analogous to that in Ref. [7]. Equation (11) is very similar to the finite-difference implementation of a two-dimensional diffusion problem, except that the range coordinate x replaces time in our case, and z_m and z_n replace the two spatial dimensions. When the equation is recast in matrix form with the $\Gamma\left(x; z_m; z_n\right)$ arranged as an $N^2 \times 1$ vector, one arrives at a linear system involving premultiplication with an $N^2 \times N^2$ matrix that is *tridiagonal with fringes*, so-called because the elements on the main (zeroth), the ± 1, and the $\pm N$ diagonals are nonzero. It is a huge matrix for any practical problem. Since the upper absorbing layer in an atmospheric parabolic equation (PE) solution is usually about 30 wavelengths thick, and the vertical spacing of the grid is typically 0.1 wavelengths, N is at least 300. For low-frequency calculations, $N = 1000$ is representative. Hence, the matrices would have dimensions $10^6 \times 10^6$; these 10^{12} elements could not be stored directly in memory on any computer. Fortunately, the fringed matrix is highly sparse: only about 5×10^6 elements would be nonzero, and it is feasible to store these nonzero elements.

Solution of such sparse systems is a well-studied problem in numerical analysis. The fringes complicate matters significantly in comparison to purely tridiagonal systems (as occur in the standard Crank-Nicolson PE formulation for the pressure field). We attempted several conventional methods for solving Eq. (11), but generally found them unsatisfactory. Eventually, we settled on a method where only the coherent part of the solution is propagated at each range step, and the incoherent part is then determined by iteration. This method will be described in more detail in an upcoming publication.

An example of a calculation for sound propagation in an upward-refracting atmosphere is shown in Fig. 1. The vertical axis corresponds to the mean sound pressure level relative to cylindrical spreading. The source frequency is 40 Hz, source height is 5 m, the receiver height is 2.6 m, and $\beta = (20.8 + i19.2)^{-1}$. The figure compares the average of 40 runs made with a standard Crank-Nicholson PE (based on the description in Ref. [7]) to the direct solution of Eq. (11). A turbulence correlation function based on the von Karman model [8,9] was used in the calculations. Both methods predict nearly identical increases of sound levels in the shadow zone due to scattering by turbulence. Equation (11) yields a much smoother prediction, however. The main disadvantage of Eq. (11), as currently implemented, is its computational time: it required about 24 hours on a 500-MHz Pentium III PC, which was 12 times longer than the total for the 40 standard PE runs.

5 - STATISTICAL MOMENTS OF ARBITRARY ORDER

A closed equation for the statistical moment of a sound field of order $n + m$ is given by:

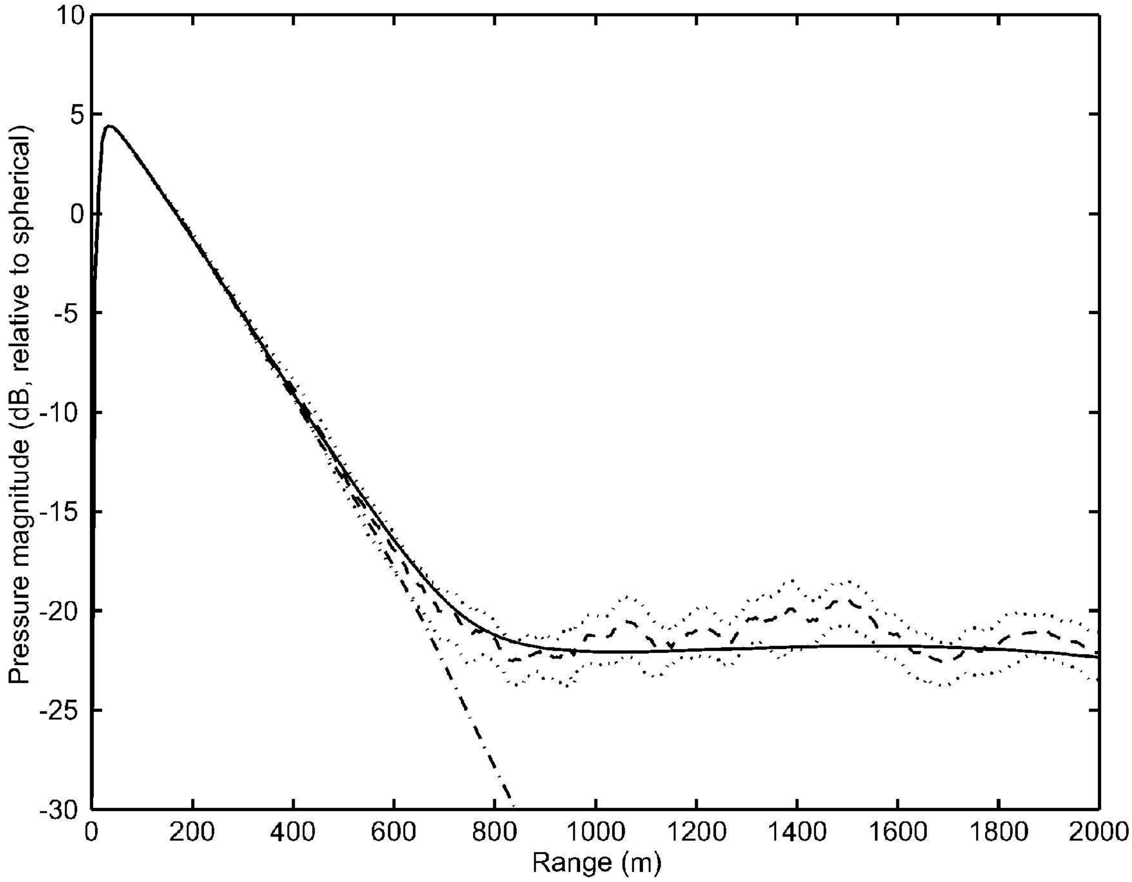

Figure 1: Comparison of mean sound pressure levels for upward refraction, calculated using various methods; dash-dotted line: calculation without turbulence; solid line: Eq. (11) with turbulence; dashed line: average of 40 runs of a standard Crank-Nicholson PE using random "snapshots"' of the turbulence; dotted line: 90% confidence intervals for the 40 runs.

$$\frac{\partial \Gamma_{n,m}}{\partial x} - \frac{i}{2k_0}\left[M\left(\vec{r}_1\right) + ... + M\left(\vec{r}_n\right) - M^*\left(\vec{r}_1'\right) - ... - M^*\left(\vec{r}_m'\right)\right]\Gamma_{n,m} + \frac{k_0^2}{8}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} b\left(x;\vec{r}_i;\vec{r}_i\right)\right.$$
$$\left. + \sum_{i=1}^{m}\sum_{j=1}^{m} b\left(x;\vec{r'}_i;\vec{r}_j'\right) - 2\sum_{i=1}^{n}\sum_{j=1}^{m} b\left(x;\vec{r}_i;\vec{r}_j'\right)\right]\Gamma_{n,m}\left(x;\vec{r}_1,...,\vec{r}_n,\vec{r}_1',...,\vec{r}_m'\right) = 0. \tag{12}$$

This equation for $\Gamma_{n,m}$ should be supplemented by the initial and boundary conditions. The former is given by:

$$\Gamma_{n,m}\left(0;\vec{r}_1,...,\vec{r}_n;\vec{r}_1',...,\vec{r}_m'\right) = \left\langle p_0\left(x,\vec{r}_1\right)...p_0\left(x,\vec{r}_n\right)p_0^*\left(x,\vec{r}_1'\right)...p_0^*\left(x,\vec{r}_m'\right)\right\rangle. \tag{13}$$

The boundary conditions formulated at $z_i = 0$ are

$$\left(\partial/\partial z_i + ik_0\beta\right)\Gamma_{n,m}\left(x;\vec{r}_1,...,\vec{r}_n;\vec{r}_1',...,\vec{r}_m'\right) = 0,\ i = 1,...,n, \tag{14}$$

and at $z_j' = 0$ they are

$$\left(\partial/\partial z_j' - ik_0\beta^*\right)\Gamma_{n,m}\left(x;\vec{r}_1,...,\vec{r}_n;\vec{r}_1',...,\vec{r}_m'\right) = 0,\ j = 1,...,m. \tag{15}$$

Note that Eqs. (5) and (8) are particular cases of Eq. (12) for $n = 1$, $m = 0$ and $n = 1$, $m = 1$, respectively. Equation (12) for $n = 2$, $m = 2$ allows one to study intensity fluctuations of a sound wave propagating in a turbulent atmosphere.

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