Practical considerations of the acoustic FEM for higher frequencies

Steffen Petersen[1], Daniel Dreyer[1, 2], Otto von Estorff[1]

[1] *Technical University Hamburg-Harburg, D-21071 Hamburg, Germany, Email: steffen.petersen@tu-harburg.de, estorff@tu-harburg.de*

[2] *now: AUDI AG, D-85045 Ingolstadt, Germany, Email: daniel.dreyer@audi.de*

Introduction

The Finite Element Method (FEM) is an indispensable tool in today's simulation tasks. It is at the core of any concept that attempts to reduce time-to-market and/or enforces virtual prototyping in almost every engineering field. However, when simulating time harmonic acoustics using the FEM, an efficient tool for bridging the "mid-frequency gap" is still missing. The lower frequency range is well covered through element-based methods, like domain- or boundary-integral based formulations. The higher frequency range, in turn, benefits from the development of energy averaging methods. However in between, the mode density is too low to render the high-frequency methods effectively, and the FEM suffers with increasing wave number from the so-called pollution effect [1].

Several approaches exist that attempt to extend the element-based methods to the mid-frequency range. These generally rely on knowledge-based concepts, i.e. incorporating a-priori-knowledge of the solution into the numerical method itself. However, these concepts appear to still lack sufficient maturity for reliable and stable simulations. In this contribution, a more straightforward approach to tackle the pollution effect is suggested, namely through still relying on the conventional FEM, but taking higher order polynomial shape approximations into account.

Pollution and Dispersion

The pollution effect is closely related to the dispersion, namely the difference of the wavelength for the numerical FEM solution and the exact wavelength at high wave numbers. For acoustic simulations at high frequencies using conventional linear or quadratic elements, this effect usually leads to uneconomical mesh sizes.

In recent years, various different approaches to control and reduce the dispersion have been developed. Although pollution may be completely avoided only in one dimension [2], these methods at times render effective also when applied to higher dimensional numerical experiments. However, most of these stabilized approaches seem to be rather complicated for simulations on general non-uniform meshes.

Shape approximation

In order to control pollution and numerical dispersion the use of higher order shape function in the conventional finite element formulation is suggested. Three different finite element families with different polynomial shape approximations are presented: Integrated Legendre polynomials [3], Bernstein polynomials, and a family of spectral element shape functions, given by Lagrange polynomials through the Gauss-Lobatto or Fekete points on quadrilateral and triangular elements, respectively [4].

The choice of appropriate shape functions strongly affects the conditioning of the overall system matrix. Regarding the simulation of large scale acoustic problems including Krylov solvers, this significantly affects the performance of the solution process. However, the rather complex geometries of most practically relevant cavities usually require a certain degree of geometric accuracy. Therefore the polynomial order p is kept at moderate levels.

Numerical Examples

The shape functions mentioned above are currently integrated in the high performance FE library libMesh [5]. The numerical examples shown here are performed considering a rectangular domain with essential and natural boundary conditions on two sides of the boundary, prescribing the sound pressure and the normal velocity v_n respectively. The remaining boundary is prescribed with $v_n = 0$. For this simple problem an analytical solution may be derived, which is used in order to monitor the accuracy of the computations. The mesh consists of either 8-noded quadrilateral QUAD8 or 6-noded triangular elements TRI6, where the location of interior nodes (not lying on the boundary) is randomly modified in order to obtain an irregular mesh. An example mesh with triangular elements is shown in Figure 1. For all computations the fluid properties for air (density $\rho = 1.225$ kg/m^3 and wave speed c = 340 m/s) are used.

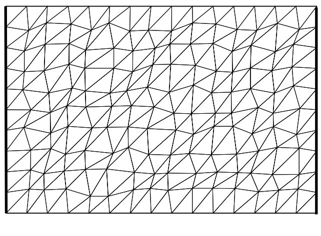

Figure 1: Model problem meshed with randomly distorted triangular elements.

In a first step, the performance of the various finite element families in combination with iterative solvers is tested. Figure 2 shows the number of iterations against the frequency for the three finite element families with polynomial order 3 and 5. For the simulations of Figure 2, a transpose-free quasi-minimal residual (TFQMR) Krylov solver is used, preconditioned with an incomplete factorization. Convergence is said to be achieved when the residual is $||r|| < 10^{-10}$. The number of triangular elements is chosen such that in all simulations the total number of degrees of freedom remains the same.

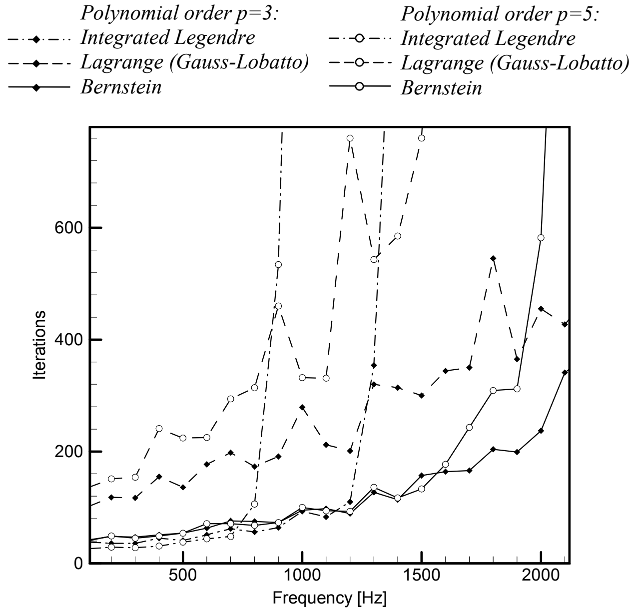

Figure 2: Iteration counts vs. frequency for different element families and different polynomial orders.

For the problem investigated here, the best performance is achieved with the finite elements based on Bernstein polynomial shape approximations. From the results in Figure 2 it may also be seen, that the performance of the iterative solvers and the stability of the solution process decreases with increasing polynomial order and increasing frequency.

The results shown in Figure 3 depict the efficiency of the *p*-FEM concept suggested, where the error of the finite element solution is plotted versus the total number of degrees of freedom.

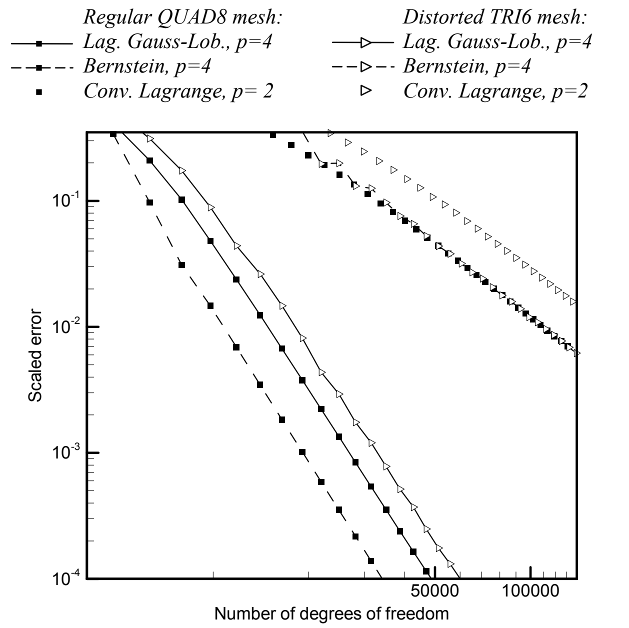

Figure 3: Scaled L_2 error vs. number of degrees of freedom at 1500 Hz.

To determine and compare the accuracy of different finite element families, a scaled error based on the L_2 error norm is adopted. The numerical results are obtained from simulations on a regular quadrilateral mesh and a distorted triangular discretization (representing the best and worst case scenario). Finite elements based on fourth order spectral (Lagrange polynomials through Gauss-Lobatto or Fekte points) and Bernstein polynomials are compared with conventional second order Lagrangian elements.

The results in Figure 3 clearly show that higher order shape functions increase the efficiency of the finite elements significantly. At this, spectral element shapes on the regular quadrilateral mesh yield the most efficient formulation. However, for practical applications, which may involve highly irregular meshes containing triangular as well as quadrilateral elements, the *p*-FEM concept based on Bernstein shapes considered here, provides remarkably high accuracy. This is true for arbitrary numbers of degrees of freedom, irrespective of the element type used.

Conclusions

Using iterative solution algorithms the Bernstein shape functions in the *p*-FEM formulation provided the most stable and efficient solution. However, applying Krylov solvers, high polynomial shape approximations decrease the stability of the solution algorithm. Hence, a reduction of the total number of degrees of freedom, stemming from high approximation orders may be nullified when iterative solution algorithms are used that may decrease in terms of computational efficiency with increasing order *p*.

The *p*-FEM concept suggested here provides an efficient method for simulating acoustics at higher frequencies, where the model sizes and respective number of degrees of freedom eliminates the use of the conventional linear FEM.

Acknowledgements

The authors would like to thank Mr Hendrik van der Heijden for his work on coding the various shape function families.

References

[1] Deraemaeker, A., Babuška, I., Bouillard, P. Dispersion and pollution of the FEM solution for the Helmholtz equation in one, two and three dimensions. Int. J. Numer. Meth. Engrg **46** (1999), 471-499.

[2] Babuška, I., Sauter, S. A. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? SIAM Journal on Numerical Analysis **34** (1997), 2392-2423.

[3] Szabo, B. A., Babuška, I. Finite Element Analysis, Wiley, Chichester, 1991.

[4] Taylor, M. A., Wingate, B. A. Vincent, R. E. An algorithm for computing Fekete points in the triangle. SIAM J. Numer. Anal. **38** (2000), 1707-1720.

[5] Kirk, B. S., Peterson, J. W., libMesh. URL: `http://libmesh.sf.net`, 2004.