Tuning of Musical Glasses through material removal

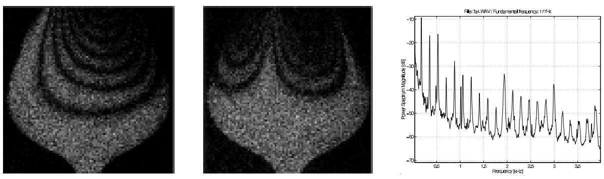
Thomas Guignard[1], Vincent Martin[2]

[1] Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Suisse, Email: thomas.guignard@epfl.ch

[2] Centre National de la Recherche Scientifique, F-78210 St. Cyr, France, Email: vmartin@ccr.jussieu.fr

Generalities and studied problem

Traditionnaly, a musical glass is shaved just over the stem to obtain the desired pitch. Can the physical cause of this traditional method be identified? In preamble, some vibrational modes of a glass and its frequency spectrum are observed.



(a) 710Hz (b) 1430Hz (c) F4 (fond. freq. 705Hz)

Figure 1: Modal forms and spectrum of a vibrating glass

It is commonly admitted that the first circonferential mode affects the first partials. The number of harmonics when the glass is excited (Figure 1(c)) and the fast disappearance of the higher orders when the glass is no longer excited are noteworthy. This spectrum cannot be fully explained as of today.

Chosen approach

In the case of a glass of very simple geometry, A.P. French [1] gives in 1983 an expression of the eigenfrequencies that puts the way to considering the tuning problem. In fact, if n is the order of the circonferential mode ($n \geq 2$) and m the order of a flexion mode of a generatrix of the cylindrical glass, the frequencies are described as

$$f_{mn} = Kh\sqrt{\frac{(n^2-1)^2 + (\beta_m b)^4}{1+\frac{1}{n^2}}} \tag{1}$$

where $\beta_m \simeq m - \frac{1}{2}$ (very roughly for $m = 1$) and $b = \frac{\pi R}{L}$ with R the radius of the cylinder, L its height and h the thickness of its walls. Rewriting

$$f_{mn} \simeq Kh\sqrt{[(n^2-1)^2 + (\beta_m b)^4](1 - \frac{1}{n^2})} \tag{2}$$

the contribution of the circonferential index n and of the flexion index m are decoupled when $\frac{1}{n^2}$ is neglected before 1 (also very roughly for $n = 2$). One would therefore obtain $f_{mn} \simeq \sqrt{f_m^2 + f_n^2}$ and actually f_m is nearly the expression of the frequency of a clamped-free beam in flexion. Moreover, the tapering of the glass' bottom seems to affect the circonferential modes (f_n^2) very little, on the contrary for f_m^2. It seems therefore natural to consider the tuning problem via the study of the influence of material removal on a generatrix of the glass, assimiliated to a clamped-free beam. The considered beam is made of glass ($E = 6.15 \cdot 10^{10}\,\text{Pa}$, $\rho = 2880\,\text{kg}\cdot\text{m}^{-3}$), has a length of 5cm and a thickness of 1.5mm.

Analytical method of small perturbations

A simple situation enlightens the flexion equation of a non-uniform beam

$$\frac{\partial^4 \zeta}{\partial y^4} + \frac{2}{I}\frac{\partial I}{\partial y}\frac{\partial^3 \zeta}{\partial y^3} + \frac{1}{I}\frac{\partial^2 I}{\partial y^2}\frac{\partial^2 \zeta}{\partial y^2} - \frac{\rho S \omega^2}{EI}\zeta(y) = 0 \tag{3}$$

Let it be a beam of length L, thickness h, rectangular section S, density ρ, Young's modulus E, gyration radius a and moment I. If the thickness varies linearly along Oy, one has $h(y) = h(0)(1+\mu y)$ or $\frac{h(L)}{(1+\mu L)}(1+\mu y)$ with $\mu L \ll 1$. Noting $h_0 = h(0)$ and $h_L = h(L)$, the expressions of the first order are: $S(y) = S_0(1+\mu y)$, $a^2(y) = a_0^2(1+2\mu y)$, $I(y) = I_0(1+3\mu y)$ and $\gamma^4(y) = \gamma_0^4(1-2\mu y)$ with $S_0 = h_0 l_x$, $a_0^2 = \frac{h_0^2}{12}$, $I_0 = \frac{h_0^2 S_0}{12}$, $\gamma_0^4 = C_0\omega^2$, $C_0 = \frac{\rho S_0}{EI_0}$, from which one obtains

$$\frac{\partial^4 \zeta}{\partial y^4} + 6\mu\frac{\partial^3 \zeta}{\partial y^3} - \gamma^4(y)\zeta(y) = 0 \tag{4}$$

The goals are the eigenfrequencies and modes. The modes $\zeta_m(y)$, searched in the base of the modes $\zeta_{k0}(y)$ of the beam of uniform thickness h_0 (or h_L) are written $\zeta_m(y) = \sum_{k=1}^{\infty} b_{mk}\zeta_{k0}(y)$. $\zeta_{m0}(y)$ outweights the rest when $b_{mm} \gg b_{mk}\ \forall\ m \neq k$. The awaited frequencies (near the frequencies of the uniform case) are written $f_m = f_{m0}(1+\theta_m)$ with $\theta_m \ll 1$, or $\omega_m^2 = \omega_{m0}^2(1+2\theta_m)$.

The resolution of the first order being the goal, using the above approximations, $\frac{\partial^4 \zeta_{m0}}{\partial y^4} = C_0\omega_{m0}^2\zeta_{m0}$ and $\frac{\partial^4 \zeta_{k0}}{\partial y^4} = C_0\omega_{k0}^2\zeta_{k0}$, one obtains

$$b_{mm}(-6\mu\frac{\partial^3 \zeta_{m0}}{\partial y^3} - 2\mu y C_0\omega_{m0}^2\zeta_{m0} + 2\theta_m C_0\omega_{m0}^2\zeta_{m0}) = \sum_{k\neq m} b_{mk}C_0(\omega_{k0}^2 - \omega_{m0}^2)\zeta_{k0} \tag{5}$$

Since θ_m only appears in the first term, a way to isolate it only would give access to θ_m. In fact, the projection of the total expression on mode $\zeta_{j0}(y)$ with $j \neq k$ and particularly for $j = m$ gives

$$\theta_m = \left[2\int_0^L y\zeta_{m0}^2\mathrm{d}y - \frac{3}{\gamma_{m0}^4}\left(\frac{\partial \zeta_{m0}}{\partial y}\right)^2_{y=L}\right]\frac{\mu}{L} = K\frac{\mu}{L} \tag{6}$$

where the modes $\zeta_{m0}(y)$ and their derivatives are analytically available [2]. In the case of material removal at the bottom of the glass, the initial situation uses a uniform thickness of h_L. The eigenfrequency depending directly on h, it appears $f_m \simeq f_{m0}\left(1+K_m\frac{\mu}{L}\right) = f_{mL}\frac{h_0}{h_L}\left(1+K_m\frac{\mu}{L}\right) \simeq f_{mL}\left[1+\left(\frac{K_m}{L^2}-1\right)\mu L\right]$.

Numerical methods

The flexion equation is

$$\frac{\partial^4\zeta}{\partial y^4}+6\mu\frac{\partial^3\zeta}{\partial y^3}-\gamma_0^4(1-2\mu y)\zeta(y)=\delta(y-y_s) \quad (7)$$

with an excitation source in y_s. The solution satisfies $\int_0^L \nu(y)\left[\frac{\partial^4\zeta}{\partial y^4}+6\mu\frac{\partial^3\zeta}{\partial y^3}-\gamma_0^4(1-2\mu y)\zeta(y)\right]\mathrm{d}y = \int_0^L \nu(y)\delta(y-y_S)\mathrm{d}y\ \forall\nu(y)$ belonging to a subspace C_ν to be determined. Taking the boundary conditions into account, integration by parts gives

$$\int_0^L \frac{\partial^2\nu}{\partial y^2}\frac{\partial^2\zeta}{\partial y^2}\mathrm{d}y-\gamma_0^4\int_0^L \nu\zeta\mathrm{d}y+$$
$$2\mu\left(-3\int_0^L\frac{\partial\nu}{\partial y}\frac{\partial^2\zeta}{\partial y^2}\mathrm{d}y+\gamma_0^4\int_0^L\nu y\zeta\mathrm{d}y\right)=\int_0^L\nu f\mathrm{d}y \quad (8)$$

With the domain $[0\ldots L]$ decomposed into elements, the integral becomes $\int_0^L \ldots dy = \sum_{j=1}^N \int_{y_j}^{y_{j+1}} \ldots dy$; the element j has nodes j and $j+1$ of coordinates y_j and y_{j+1}.

On each element, the displacement is sought under a polynomial cubic form. To the element j on $[y_j, y_{j+1}]$ corresponds the segment $[0, y_{j+1}-y_j]$ via $y=y_j+u$, resulting in $\int_{y_j}^{y_{j+1}} f(y)dy = \int_0^{h_j} f(u)du$. The form of $\zeta(u)$ on the element $[0,h_j]$ leads to $\zeta(u)=N_1(u)\zeta_j+N_2(u)\theta_j+N_3(u)\zeta_{j+1}+N_4(u)\theta_{j+1}$ with the base functions$N_1(u)=\frac{2}{h_j^3}u^3-\frac{3}{h_j^2}u^2+1$, $N_2(u)=\frac{1}{h_j^2}u^3-\frac{2}{h_j}u^2+u$, $N_3(u)=-\frac{2}{h_j^3}u^3+\frac{3}{h_j^2}u^2$ and $N_4(u)=\frac{1}{h_j^2}u^3-\frac{1}{h_j}u^2$.

When all elements are of the same dimension h, the matrices coming from the integration on each element lead to elementary matrix $[W_e(\omega,\mu,y_j)]$ in

$$\langle\delta\zeta_j,\delta\theta_j,\delta\zeta_{j+1},\delta\theta_{j+1}\rangle[W_e(\omega,\mu,y_j)]\begin{Bmatrix}\zeta_j\\\theta_j\\\zeta_{j+1}\\\theta_{j+1}\end{Bmatrix}$$
$$=\langle\delta\zeta_j,\delta\theta_j,\delta\zeta_{j+1},\delta\theta_{j+1}\rangle\{f_e\} \quad (9)$$

The assembly is the matrix representation of the summation of the elements and the continuity of the displacement and rotation at the nodes. When the essential boundary conditions are taken into account, the final matrix is of dimension $(2N,2N)$.

The resonance frequencies of the beam are obtained by sweeping: a displacement excitation is applied at the free end and the frequencies obtained show vibration amplitude maxima (also maxima of $\sum_{\text{nodes}}|\zeta_j|$). In the case of a beam with a thickness variation towards the clamped end, the method described above leads to a numerical solution that can be compared to the solution obtained anlytically with the small perturbation method (see Figure 2).

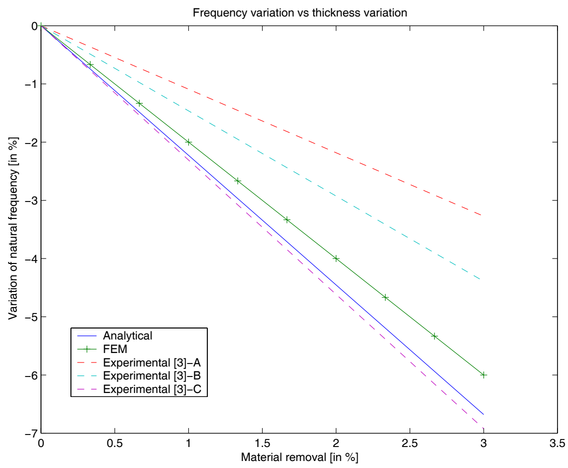


Figure 2: Variation of the first mode against material removal. The three experimental values are for different glass shapes.

Discussion

Keeping in mind that the eigenfrequency variation due to the behaviour of the glass generatrix is only a contribution to the eigenfrequence variation of the whole glass, it can nevertheless be stated that the grinding process results in a frequency reduction. Figure 2 shows that a material removal of at least 3% is necessary to reduce the frequency by a half tone (approx. 6% relative variation).

Tradition states that the frequency can be raised by more than one octave by thinning the walls near the top of the glass (here this would mean a negative μ with an initial thickness of h_0). This can not however be verified with this model, as it would provide only a very slight increase.

It is interesting to compare these frequency variations to the experimental values published in [3], which take the whole glass into account. The results obtained hint that the flexion modes are preponderant in the tuning process.

References

[1] A.P. French. In Vino Veritas: a study of wineglass acoustics. *American Journal of Physics* **51**(8): 688-694, 1983

[2] Ph.M. Morse. *Vibration and Sound*, Chapter 4, pages 151-166. The American Institute of Physics, 1983

[3] K. Oku, A. Yarai and T. Nakanishi. A new tuning method for glass harp based on a vibration analysis that uses a finite element method. *Journal of the Acoustical Society of Japan* **21**(2): 97-104, 2000

The authors wish to thank Mrs I. Emge and Mr. J.-F. Zürcher for motivating them to work on the problematic of the tuning of musical glasses.