A single-run method for evaluation of sets of mode solutions in lined ducts

F.P. Mechel
D-71120 Grafenau

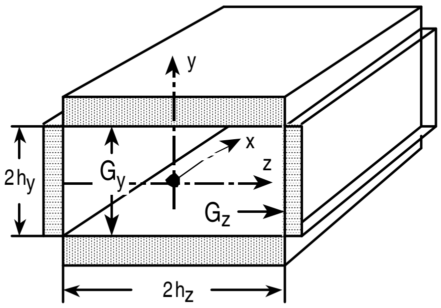
Introduction :

Modes are elementary solutions of the wave equation satisfying conditions of symmetry and boundary conditions at the duct walls. These conditions lead to the characteristic equation having the mode wave numbers as solutions. The sound field evaluation in inhomogeneous ducts or for non-homogeneous sources requires sets of mode solutions which must be • precise (nulling the char. equation), • complete (no important mode missing), • unambiguous (no mode appears repeatedly in the set). The last requirement is the most important; it implies, in conventional methods of numerical solution of the char. equation, the knowledge of precise starters, and mode range limits.

The proposed method is described below for a flat duct with locally reacting lining without flow, although it works also in round ducts, with bulk reacting linings, and with steady flow (see forthcoming paper in acta acustica).

The method is easily programmed, it computes fast and precisely up to reasonably high mode orders, and is mode-safe. It does not require start solutions, nor mode range limits.

Fundamentals :



G= lining surface admittance; time factor $e^{j\omega t}$

General mode form:

$$p(x,y,z) = q_y(\varepsilon_y y)\cdot q_z(\varepsilon_z z)\cdot e^{-\Gamma x}$$

Mode symmetry (similar in z-direction) :

$$q_y(\varepsilon_y y) = \begin{cases} \cos(\varepsilon_y y) & ; \text{ symmetrical mode} \\ \sin(\varepsilon_y y) & ; \text{ anti-symmetrical mode} \end{cases}$$

Secular equation (from wave equation):

$$\Gamma^2 = \varepsilon_y^2 + \varepsilon_z^2 - k_0^2 \quad ; \quad k_0 = \omega / c_0$$

Characteristic equation (similar in z-direction) :

$$\varepsilon_y h_y \cdot \tan(\varepsilon_y h_y) = j k_0 h_y \cdot Z_0 G_y =: j U_y \quad ; \text{symmetrical}$$

$$\varepsilon_y h_y \cdot \cot(\varepsilon_y h_y) = -j k_0 h_y \cdot Z_0 G_y =: -j U_y \quad ; \text{anti-symm.}$$

Generalized form, with

$z \equiv \varepsilon h$ searched, and $U \equiv k_0 h \cdot Z_0 G$ given quantities :

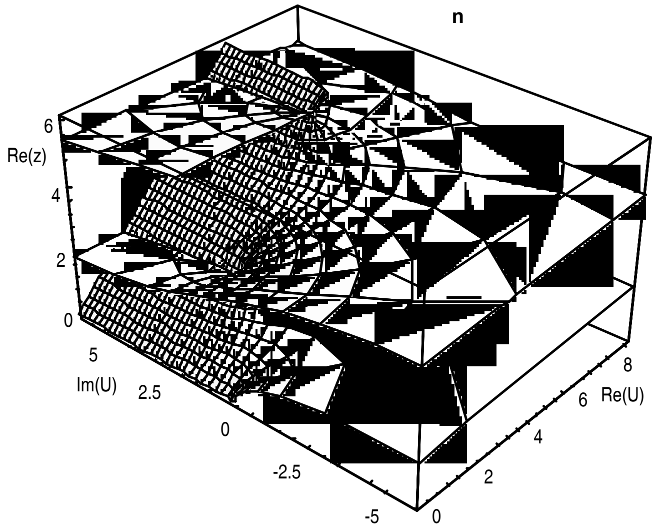
$$f(g(z);U) = z \cdot \tan(z) - jU \overset{!}{=} 0 \quad ; \text{ symmetrical mode}$$

$$f(g(z);U) = z \cdot \cot(z) + jU \overset{!}{=} 0 \quad ; \text{ anti-symmetrical mode}$$

Thus :

$$g_{sy}(z) = z \cdot \tan(z)$$
$$g_{as}(z) = z \cdot \cot(z) = z^2 / (z \cdot \tan(z)) = z^2 / g_{sy}(z)$$

3D-mode chart for symmetrical modes (with curves Re(z)= const, and Im(z)=const over the complex U-plane):



A method of mode set evaluation must unambiguously subdivide this Riemann surface into single-mode leaves. The branch cuts must start in the branch points; the "ramp" of the surface wave mode will be distributed into the other modes.

Continued-fraction expansion :

The transcendental and quasi-periodic functions g(z) of the char.equation can be expanded as continued-fractions (cf), which vonverge in the whole z plane (except in poles).

$$g(z) = z \cdot \tan z = \frac{z^2}{1-}\,\frac{z^2}{3-}\,\frac{z^2}{5-}\ldots\frac{z^2}{(2k+1)}\ldots \equiv \cfrac{z^2}{1-\cfrac{z^2}{3-\cfrac{z^2}{5-\ldots}}}$$

Numerical convergence begins at a depth k of expansion when the sub-fractions are much smaller than the leading integers of the denominators:

$$\frac{|z|^2}{(2k+1)} << (2k-1) \quad \Leftrightarrow \quad (2k)^2 - 1 >> |z|^2$$

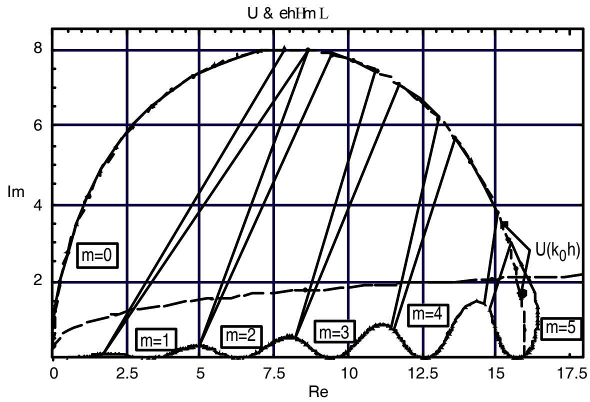
The transcendental characteristic equation

$$f(g(z);U) = \frac{z^2}{1-}\,\frac{z^2}{3-}\,\frac{z^2}{5-}\ldots\frac{z^2}{(2k+1)} - jU \overset{!}{=} 0$$

can be transformed by a truncated continued-fraction, after collecting the truncated cf in a single fraction, $cf(z^2) = Num(z^2)/Den(z^2)$, into a polynomial equation of z^2

$$p(z^2;U;k) := Num(z^2) - jU\cdot Den(z^2) \overset{!}{=} 0$$

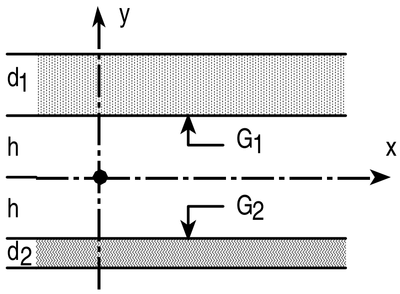
Solving polynomial equations is a standard task of numerical mathematics; fast and precisely computing programs exist, not needing start solutions. If the depth k of the cf-expansion is sufficiently large, and the polynomial solutions are ordered in a list with increasing real parts, the usable mode solutions z up to a mode order limit mhi will be found at the beginning of that list $\{z_0, z_1, z_2, \ldots, z_{mhi}, \ldots\}$. For a rectangular duct with locally reacting lining, the relation between k and mhi≤15 is k= max(4·mhi+1 , 45).



Frequency response curves of the absorber function $U(k_0h)$ of high-Q resonators and of solutions $\varepsilon h(m)$ for symmetrical modes with orders m= 0,1,…,5 in a rectangular duct. (the flat, dashed curve connects the branch points).

The surface wave mode is distributed over the other modes.

More complicated ducts :



A rectangular duct with different linings on opposite sides.

Absorber functions:

$$U_i = k_0h\cdot Z_0G_i \quad ; \quad i=1,2$$

$$U_{sy} := \frac{1}{2}(U_1 + U_2) \quad ; \quad U_{as} := \frac{1}{2}(U_1 - U_2)$$

Mode form :

$$p(x,y) = (A\cdot\cos(\varepsilon y) + B\cdot\sin(\varepsilon y))\cdot e^{-\Gamma x}$$

$$(\Gamma h)^2 = (\varepsilon h)^2 - (k_0h)^2$$

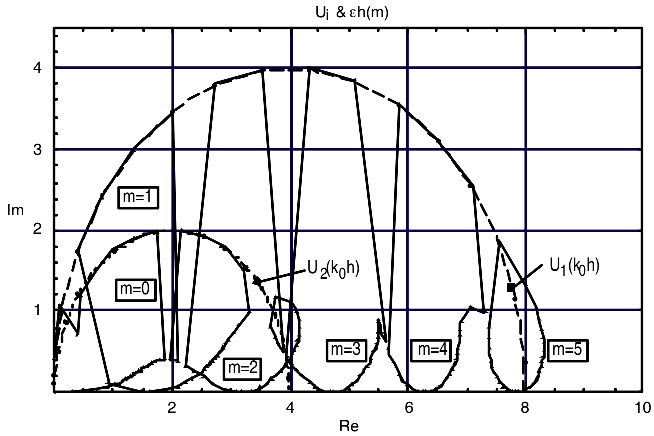
$$\frac{B}{A} = -\cot(\varepsilon h)\frac{\varepsilon h\cdot\tan(\varepsilon h) - jk_0h\cdot Z_0G_2}{\varepsilon h\cdot\tan(\varepsilon h) + jk_0h\cdot Z_0G_2}$$

The characteristic equation (with $z \equiv \varepsilon h$) reads:

$$[z\cdot\tan z - jU_{sy}]\cdot[z\cdot\cot z + jU_{sy}] = U_{as}^2$$

or equivalently $f(g_{sy}(z); U_{sy})\cdot f(g_{as}(z); U_{sy}) = U_{as}^2$

Obviously it can be transformed with continued fractions to a polynomial equation in z^2 .

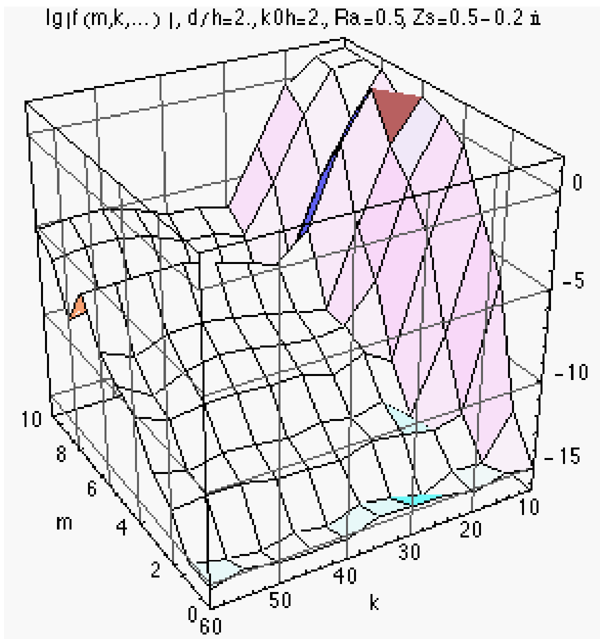


Frequency response curves of the absorber functions $U_i(k_0h)$ of resonators with different tuning and quality on opposite duct sides in a rectangular duct, and of solutions $\varepsilon h(m)$ with orders m= 0,1,…,5 .

Mode-safe evaluation of this mode set by conventional methods would be difficult.

Numerical tests :

The presented method was tested in several aspects (see paper in acta acustica). Of great interest is the relation between the depth k of cf-expansion and the range of mode orders m obtained. The last graph shows that relation for a bulk reacting lining of a porous layer (thickness d ; norm. flow resistance R_a) which is covered with a resistive foil (norm.partition impedance Z_s). The decadic logarithm of the magnitude of the char.equation with polynomial solutions inserted is plotted over the mode order m and the expansion depth k.



An expansion depth k≥40 should be applied.