Hydrodynamic Pressure on Multiple Vertical Cylinders in a Compressible Fluid

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Introduction

The numerical study of the acoustic interaction among multiple obstacles in a compressible fluid is a challenging task, especially if the number of interacting bodies is large, the fluid region extends to infinity, and the wave numbers of interest are high. The acoustic pressure on such structure assemblies can effectively be predicted by using the so-called "exact algebraic method". The method was originally developed by Kagemoto and Yue [1] for the calculation of hydrodynamic loads due to gravity water waves on large floating offshore structures. In the present work, the method has been extended and adopted to acoustic scattering and radiation problems. Although the computational example presented in this paper refers to the field of ocean and coastal engineering, the applications in other fields, where the acoustic scattering and radiation is of primary concern, are certainly possible.

Mathematical formulation of the problem

Consider the scattering and radiation of acoustic waves by a finite array of N fixed vertical cylinders in a layer (depth h) of a compressible fluid (water) with a free surface. The cylinders, not necessarily circular, extend throughout the water depth. Otherwise, the fluid region is not bounded. The origin of a fixed reference frame *(x,y,z)* is on the fluid bed and the z-axis points upwards. There are $N+1$ polar coordinate systems in the *(x,y)*-plane: (r,θ) centered at the origin and (r_j,θ_j), $j=1,...N$, centered at (x_j,y_j), the center of the j^{th} cylinder. The various parameters relating to the relative positions and size of the cylinders are shown in **Figure 1**.

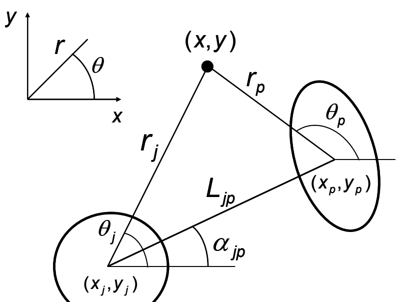

Figure 1: Plane view of two cylinders.

The acoustic scattering and radiation is governed by:
- the wave equation for pressure p in the fluid domain,
- the linearized boundary conditions on the free surface $z=h$ and at the impermeable bed $z=0$

$$p=0 \quad \text{on } z=h\,, \quad \frac{\partial p}{\partial z}=0 \quad \text{on } z=0 \tag{1}$$

- the boundary conditions on the wetted cylinder surface

$$\frac{\partial p}{\partial n_j}=0\,, \quad \text{or} \quad \frac{\partial p}{\partial n_j}=-\rho a_{j,n}\,, \quad j=1,...,N \tag{2}$$

for the acoustic scattering and radiation, respectively. $a_{j,n}$ denotes the normal component of the acceleration on the wetted surface of the j^{th} cylinder and ρ stands for the density of the fluid. For the time-harmonic acoustic waves, the complex-valued pressure *P(x,y,z)* may be introduced by writing

$$p(x,y,z,t)=\mathrm{Re}\left\{P(x,y,z)e^{-i\omega t}\right\}. \tag{3}$$

Under the absence of both the incident waves and the motion of the structure, the z-dependence of the *eigenfunctions* of the boundary value problem (1) can be factored out according to

$$P_m(x,y,z)=\Psi(x,y,k_m)\cdot\cos(\kappa_m z)\,, \tag{4}$$

where

$$\kappa_m h=\frac{\pi}{2}+m\pi\,,\ m=0,1,..\ \text{and}\ k_m^2=\frac{\omega^2}{c^2}-\kappa_m^2\,. \tag{5}$$

The latter relation defines the cut-off frequency of the wave propagation in the fluid layer $\omega_{cut}=\pi c/2h$ where c is the speed of sound inside the fluid.

Exact algebraic method

For the sake of convenience, the mathematical formulation of the exact algebraic method will be presented for a simultaneous scattering and radiation problem. Thus, it will be assumed that an acoustic wave incident on the cylinders making an angle β with the positive x-axis is given by

$$p_0(r,\theta,z,t,k_0)=A\cos(\kappa_0 z)e^{i[k_0 r\cos(\theta-\beta)-\omega t]} \tag{6}$$

and that each cylinder is subjected to horizontal oscillations according to $u(t)=u_0 e^{-i\omega t}$. Obviously, one can construct other incident waves satisfying the boundary conditions (1) as a superposition of elementary waves $p_0(r,\theta,z,t,k_m)$.

Kagemoto and Yue (1986) showed in the context of water gravity waves that an interaction theory can be developed for an array of structures which have the property that they are 'vertically separated'. This means that the vertical projections of the structures must not intersect. Moreover, the escribed vertical circular cylinder to each structure centered at its respective origin must not enclose any other origin. Under these assumptions the complex amplitude of an outgoing cylindrical wave emanating from the j^{th} structure can be given by a truncated series

$$P_j(r_j,\theta_j,z)=\sum_{m=0}^{M}Z_m(z)\sum_{n=-\infty}^{N_m}A_{mn}^j H_n^{(1)}(k_m r_j)e^{in\theta_j}\,, \tag{7}$$

where the depth eigenfunctions are given by (4), k_m by (5), and A_{mn}^j are the coefficients to be determined. It should be noted that the Hankel functions $H_n^{(1)}(k_m r_j)$,which describe the outgoing waves, should be replaced by the modified Bessel functions $K_n(\tilde{k}_m r_j)$ as soon as the wave numbers k_m defined by (5) become complex, $k_m=i\tilde{k}_m$, for

$\kappa_m^2 > \omega^2 / c^2$. The corresponding pressure components represent evanescent modes which decay exponentially away from the structure. The relation (7) can concisely be written in a matrix form

$$P_j(r_j,\theta_j,z) = \mathbf{A}_j^{\mathbf{T}} \boldsymbol{\Psi}_j(r_j,\theta_j,z), \qquad (8)$$

where $\mathbf{A}_j^{\mathbf{T}}$ is the vector of coefficients A_{mn}^j and $\boldsymbol{\Psi}_j$ is the vector of scattered partial cylindrical waves $Z_m(z)H_n^{(1)}(k_m r_j)e^{in\theta_j}$ .The total incident pressure P_0^p upon a structure *p* will consist of the incident wave (6), which in a series representation reads

$$P_0 = AZ_0(z)e^{ik_0(x_p \cos\beta + y_p \sin\beta)} \sum_{n=-\infty}^{n=+\infty} i^n J_n(k_0 r_p) e^{in\theta_p}, \qquad (9)$$

the scattered waves given by (8) and the waves radiated by the isolated structures *j* due to their motion

$$P_j^R(r_j,\theta_j,z) = \sum_{m=0}^{M} Z_m(z) \sum_{n=-\infty}^{N_m} C_{mn}^j H_n^{(1)}(k_m r_j) e^{in\theta_j}, \qquad (10)$$

where the constants C_{mn}^j can directly be determined from the boundary conditions (2). One obtains

$$P_0^p = P_0 + \sum_{j=1,\ j\neq p}^{N} P_j^R + \sum_{j=1,\ j\neq p}^{N} \mathbf{A}_j^{\mathbf{T}} \boldsymbol{\Psi}_j . \qquad (11)$$

Using Graf's addition theorem for Bessel functions [2] one can express all scattered partial cylindrical waves in equations (8) and (10) by the local, with respect to the structure *p*, 'incident' partial cylindrical waves of the form $Z_m(z)J_n(k_m r_p)e^{in\theta_p}$. It follows that, for $j,p = 1,\dots,N,\ j \neq p$,

$$H_n^{(1)}(k_m r_j)e^{in\theta_j} = \sum_{q=-\infty}^{+\infty} H_{n-q}^{(1)}(k_m L_{jp}) e^{i(n-q)\alpha_{jp}} J_q(k_m r_p) e^{iq\theta_p} \qquad (12)$$

A similar relation holds for evanescent wave components. Using a suitably truncated version of (12) one obtains

$$\boldsymbol{\Psi}_j = \mathbf{T}_{\mathbf{jp}} \boldsymbol{\Phi}_p \qquad (13)$$

where $\boldsymbol{\Phi}_p$ is a vector of incident cylindrical waves and the elements of the matrix $\mathbf{T}_{\mathbf{jp}}$ for propagating and evanescent modes are given by

$$\left[\mathbf{T}_{\mathbf{jp}}\right]_{nq} = H_{n-q}^{(1)}(k_m L_{jp}) e^{i(n-q)\alpha_{jp}},$$
$$\left[\mathbf{T}_{\mathbf{jp}}\right]_{nq} = (-1)^q K_{n-q}(\tilde{\kappa}_m L_{jp}) e^{i(n-q)\alpha_{jp}}.$$

With the use of (13), the total incident pressure upon the structure *p* can be evaluated as

$$P_0^p = [\mathbf{b}_{\mathbf{p}}^{\mathbf{T}} + \sum_{j=1,\ j\neq p}^{N} (\mathbf{C}_{\mathbf{j}}^{\mathbf{T}} + \mathbf{A}_{\mathbf{j}}^{\mathbf{T}}) \cdot \mathbf{T}_{\mathbf{jp}}] \boldsymbol{\Phi}_p, \qquad (14)$$

where $\mathbf{b}_{\mathbf{p}}$ is the vector of coefficients of the partial wave decomposition of the incident wave (9). In general, it is possible to relate the incident and scattered pressure fields at the p^{th} structure through the diffraction characteristics of that structure in isolation. Thus, there exist 'diffraction transfer matrices' (see Martin [3]) $\mathbf{B}_{\mathbf{p}}$, $p = 1,\dots,N$, such that

$$\mathbf{A}_{\mathbf{p}} = \mathbf{B}_{\mathbf{p}} [\mathbf{b}_{\mathbf{p}}^{\mathbf{T}} + \sum_{j=1,\ j\neq p}^{N} (\mathbf{C}_{\mathbf{j}}^{\mathbf{T}} + \mathbf{A}_{\mathbf{j}}^{\mathbf{T}}) \mathbf{T}_{\mathbf{jp}}]^T \qquad (15)$$

Specifically, the element $\left[\mathbf{B}_{\mathbf{p}}\right]_{nq}$ is the amplitude of the n^{th} partial wave of the scattered pressure field due to a wave of mode *q* incident on structure *p* in isolation. The system of linear algebraic equations (15) can be solved for the unknown interaction coefficients $\mathbf{A}_{\mathbf{p}}$, $p = 1,\dots N$ provided the diffraction matrices have been determined. For general geometries the diffraction matrices must be calculated numerically.

Numerical example

Consider scattering of an acoustic wave (6) by an array of four vertical circular cylinders (radius *a*) arranged in a square mounted on the sea bed in water of constant depth *h=a*. The incident wave makes an angle $\beta = 45°$ with the *x*-axis. **Figure 2** shows the results of computations of the exciting force in the direction of wave advance (the magnitude of the force on cylinder 4 is identical to that on cylinder 2). The values have been non-dimensionalized by the forces that would be experienced if the cylinders were in isolation, so the curves represent the effects of the interaction.

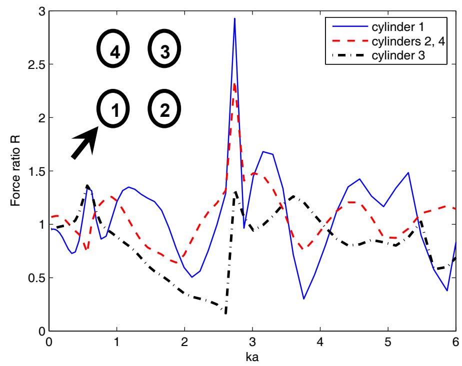

Figure 2: Dimensionless exciting forces on a group of cylinders.

Conclusions

An exact algebraic method developed originally for water-wave interaction problems has successfully been applied in the field of acoustic scattering and radiation. Using this methodology, the coefficients of acoustic interaction for arrays of structures can be determined in an exact and efficient way, provided the diffraction matrices for the structures in isolation have been determined.

References

[1] H. Kagemoto, D.K.P. Yue: Interactions among multiple three-dimensional bodies in water waves: an exact algebraic method. *J. Fluid Mech*. **166**,(1986), 189-209.

[2] M. Abramowitz, I.A. Stegun: *Handbook of Mathematical Functions*, Dover Publications, New York, (1965).

[3] P.A. Martin: On the T-matrix for water-wave scattering problems. *Wave Motion* **7**, (1985), 177-193.