Localization of scars in the stadium billiard using reassigned Husimi distribution

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Introduction

The study of eigenfunctions of quantum systems, in particular their dependence on the classical dynamics, has attracted a lot of attention over the last years. The semi-classical approximations of quantum mechanics, provides a link between classical orbits and quantum interferences. In acoustic language, this corresponds to the geometrical ray asymptotics regime, described by the eikonal equation, and the wave behaviour at high frequency regime given by the Helmholtz equation. A prominent class of applications is provided by two-dimensional billiard systems. This paper treats the case of a billiard having the shape of a stadium. The Bunimovich stadium [1, 2] has ray dynamics completely chaotic. In wave theory, the eingenfunctions often appear to have complicated structure. However, some subsets of the eigenfunctions appear more regular and can often be correlated with the patterns of short unstable periodic orbits of the classical system which are called scars [3]. In order to study how ray motions manifest themselves in wave theory, we begin by introduced Birkhoff analysis [4]. This method proposed in quantum mechanics (with Dirichlet boundary conditions) consists in taking all the information of a wave field in its normal derivatives evaluated on the boundary. Phase space distribution used on such a signal is provided by the Birkhoff variables $(s, \sin\theta)$; The variables parameterizing the surface are s, a line which defines the boundary of the problem, and its conjugated normalized wavenumber $\frac{k_{//}}{k} = \sin\theta$, which is the normalized component of the wavenumber k parallel to the wall. The mapping of Birkhoff variables in this coordinate system is called the quantum Poincaré section [4]. The wave-ray correspondence is realized by the Husimi distribution [5] introduced in quantum mechanics also called spectrogram in signal processing, and is used in the case of chaos [6, 7]. In addition, a new method called the reassignment, introduced by Kodera et al [8] in signal processing for speech analysis, improves the results of the HD by compensating for it faults. New and fast implementation of this method that makes it an attractive tool, was recently proposed by Flandrin and Auger [9].

This paper extends the reassigned approach in the semi-ray study of billiards and is organized as follows. After introducing the necessary object of the calculation of an eigenstate in a stadium billiard and some informations about phase space distribution and reassigned method, we compute scars and analyze how the reassigned HD is used to define scar measures.

Review of theory

The boundary of the stadium consists of a circle of radius R, which is split in half and slid apart a distance ϵ. The openings are filled with straight segments (Fig. 1(a)). Thus, the perimeter of the stadium is $\ell = 2\epsilon + 2\pi R$. We consider the Helmholtz equation in the two-dimensional stadium bounded region with Neumann boundary condition:

$$\begin{cases} (\Delta + k^2)\psi(\mathbf{r}) = 0, & \forall\ \mathbf{r} \in \mathcal{D}, \\ \partial_n \psi = 0, & \forall\ \mathbf{r} \in \mathcal{S}, \end{cases} \tag{1}$$

where $\mathcal{S}$ being the surface enclosing the stadium region $\mathcal{D}$ and ∂_n the normal outwardly directed component of the spatial derivative on the walls. The connection between this wave problem and a classical dynamical system is provided by the eikonal approximation. The characteristic trajectories, or rays, of this dynamical system are simply straight lines with specular reflection from the boundary. As shown in Fig. 1(b), a single typical trajectory appears to cover the entire interior of the stadium. There are however special trajectories which are not chaotic. These are periodic orbits or scars which are unstable to perturbations in initial conditions. An example is given in Fig. 1(c) which belongs to the whispering gallery (WG) family.

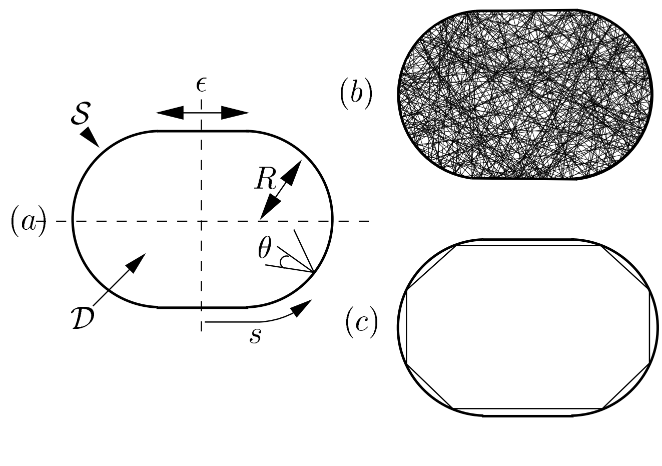

Figure 1: (a) Stadium boundary for the helmholtz equation with axis symmetry. (b) Typical example of a single trajectory. (c) Example of a scar in the stadium billiard called a whispering gallery.

In order to analyse this phenomenon in the semi-ray regime, a natural way of reducing the quantum problem from two to one dimension is to study in phase space the scar function on the boundary with Birkhoff coordinates (s, k_s). The coordinates are s, the arc length along the boundary, and its conjugate normalized wavenumber k_s, which is the component of wavenumber parallel to the wall. The HD is the most widely used tool for analyzing the local spectrum of a surface scar function. It is expressed as

$$\rho_\psi^H(s,k_s) = |GWFT_\psi(s,k_s)|^2 = \left|\int \psi(s')w^*(s-s')e^{-jk_s\cdot s'}ds'\right|^2, \quad (2)$$

where $GWFT_\psi$ denotes the gaussian windowed Fourier transform of the eigenstate $\psi(s)$, and s is the position of the center of the gaussian window $w(s)$. Indeed, instead of using its usual definition, as in Eq. 2, the HD can be equivalently expressed as

$$\rho_\psi^H(s,k_s) = \int\int \rho_\psi^W(\eta,\xi)\rho_w^W(\eta-s,\xi-k_s)\frac{d\eta d\xi}{2\pi}, \quad (3)$$

where the HD results from the smoothing of ρ_ψ^W which is the Wigner distribution (WD) [10] of the eigenstate $\psi(s)$, by ρ_w^W the WD of the window $w(s)$. Conceptually, reassignment of the HD can be considered as a squeezing, whose effect is to refocus the contributions that survived the smoothing. Thus , a normal mode of the stadium in some sense corresponds to a scar signature will be studied in phase space with and without the reassignment method.

Numerical results

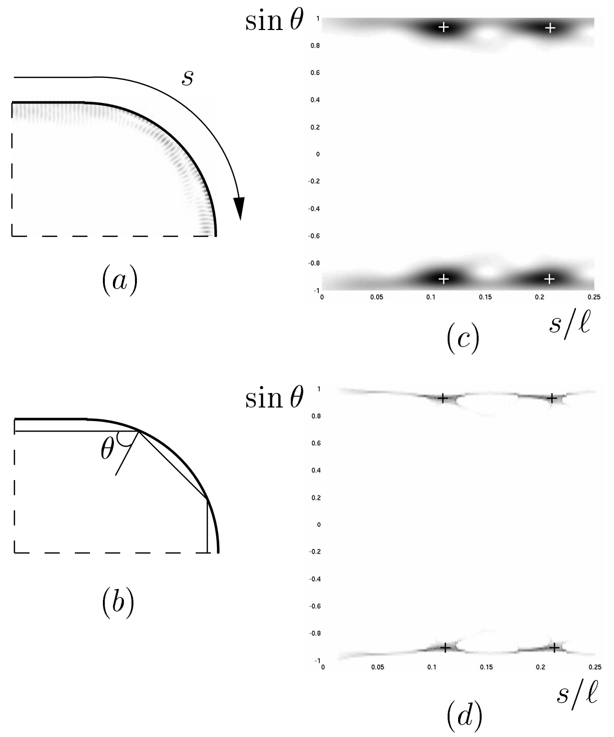

Figure 2: Representative eigenfunction of the quarter stadium in coordinate space (a), and in the Birkhoff map with the HD (c) and the reassigned HD (d).

It is common in studies of the stadium to reduce the domain using time reversal and translation symmetry to obtain the quarter-stadium. The axis symmetry are shown in fig. 1(a). We have numerically solved the wave problem using a finite elements method. Fig. 2(a) presents results obtained for the quarter-stadium in the range of $k = 105.3815$. First, the correspondence between the eigenstate and the nature of ray dynamics is clearly visible in Fig. 2(a) and (b). In Fig. 2(c) we computed the HD for the WG mode, and marked the ray dynamics by crosses in the corresponding quantum Poincaré section such that we can directly compare the phase space representations of waves and rays. Good agreement is noted between them. We can easily conclude that the rassigned HD (Fig. 2(d)) produces an almost ideally concentrated representation around periodic orbits. It confirms that in the HD, the energy is poorly localized, while in the reassigned HD the energy is perfectly localized. It is of interest, for the reassigned HD, to analyse the neighborhood of periodic orbits using the stable and unstable manifolds as shown in previous works for the HD [11].

Conclusion

In this work, we have investigated the ray and wave properties by applying methods known from the classical and quantum theories. Using the stadium billiard as an example, we have shown this concept to be very fruitful. We find a WG orbit associated with islands in phase space. The reassigned method gives results in good agreement with ray dynamics. Then the application of the ray-wave correspondence in sophisticated acoustical systems therefore may provides a powerful tool for future works in varying cross-section waveguides.

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