Symmetry of Nonlinear Acoustics Equations Using Group Theoretic Methods: a Signal Processing Tool for Extracting Judicious Physical Variables

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Introduction of KZ equation

Lie group methods as been recently applied to the nonlinear dynamics of gaz bubbles in liquid[1] and to Burger's equation by Abd-el-Malek[2] using the Moran method of reduction[3] with the aim of extracting physical information. The aim of our paper is to find a generalization of transformations (such as Hopf-Cole for Burgers equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0$) which originate from the KZ Lie symmetries, and to show that the Tjotta transformation[4], which has a difficult physical interpretation, is also the consequence of the invariance symmetry of KZ equation.

The propagation of a nonlinear ultrasonic wave, generated by an axisymmetric transducer of geometrical radius a, in a non dissipative medium (Fig.1) is described by the KZ equation:

$$KZ(r,\theta,z) \equiv Nu^2_{\theta\theta} - u_{\theta z} + u_{rr} + \frac{1}{r}u_r = 0, \qquad (1)$$

where $u = \frac{p'}{u_0}$ is the normalized sound pressure, $z = \frac{x}{4l_d}$ is the axial distance, $l_d = \frac{ka^2}{2}$ is the diffraction Rayleigh length, $\theta = \omega(t - x/c_0)$, is the normalized time, $l_s = \frac{c_0^2}{\epsilon\omega u_0}$ the shock-formation length, c_0 is the equilibrium sound velocity, and ϵ is the nonlinear parameter of the medium and $N = \frac{2l_d}{l_s}$.

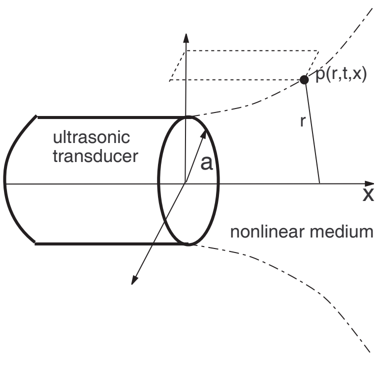

Figure 1: Nonlinear propagation of acoustic waves generated by an axisymmetric transducer of radius a

An exact solution of KZ can be written as [5]:

$$\begin{aligned} u(r,\theta,z) &= \frac{c}{1-4iz(1-ib)} \qquad (2)\\ &\times \exp\left[iT(r,\theta,z) - (1-ib)\frac{r^2}{1-4iz(1-ib)}\right], \\ T(r,\theta,z) &= \theta + Nu(r,\theta,z) \qquad (3)\\ &\times \frac{1-4iz(1-ib)}{-2i(1-ib)}\ln\left(1-4iz(1-ib)\right) \end{aligned}$$

where $T(r,\theta,z)$ is the so called "nonlinearly retarded time", b and c constants.

Symmetries of KZ equation

Let us apply the Moran[3] method of reduction to KZ:

$$KZ(u, u_i, u_{ij}, r, \theta, z) = 0 \text{ where } i, j \in \{r, \theta, z\}. \qquad (4)$$

One build a one parameter (a) group transformation $\overline{S} = C^s(a)S + K^s(a)$, where the letter S is related to all variables and functions r, θ, z, u, and $C^s(a)$, $K^s(a)$ are group functions. The derivative transformations give $\overline{S_{\bar{i}}} = C^s/C^iS_i$, and $\overline{S_{\bar{i}\bar{j}}} = C^s/C^iC^jS_{ij}$.Then, any equation $\mathcal{E}(S, S_i, S_{ij})$ of variables S and its derivatives S_i, S_{ij}, ... is said to be invariant if we have

$$\mathcal{E}(\overline{S}, \overline{S_{\bar{i}}}, S_{\overline{ij}}) = H[a]\mathcal{E}(S, S_i, S_{ij}), \qquad (5)$$

where $H[a]$ is a function of a. A basic group theory theorem [3] states that a function $\eta(r,z,u)$ is an absolute invariant of a one-parameter group if it satisfies the linear equation :

$$\sum_{i=1}^{6}(\alpha_i S_i + \beta_i)\frac{\partial\eta}{\partial S_i} = 0, \qquad (6)$$

where $\alpha_i = \frac{\partial C^{S_i}}{\partial a}(a_0)$ and $\beta_i = \frac{\partial K^{S_i}}{\partial a}(a_0)$. Generally, this new transformation of variables simplifies this equation and allows to extract judicious physical variables.

Pseudo-separation of the longitudinal variable z

Let us suppose that $u[r,z,\theta] = f[z]g[r,z,\theta]$. Expanding all partial derivatives in f and g variables, we can obtain a new KZ equation:

$$\begin{aligned} KZ(r,\theta,z) \quad &\equiv \tfrac{f[z]g_r}{r} + f[z]g_{rr} - f[z]g_{z\theta} - g_\theta f'[z] \\ &+N(2f[z]^2g_\theta^2 + 2f[z]^2g[r,z,\theta]g_{\theta\theta}) = 0. \end{aligned} \qquad (7)$$

The group invariance (5) applied to this new KZ equation and gives $\overline{KZ}(\overline{f}, \overline{g}, \overline{r}, \overline{\theta}, \overline{z}) = H[a]KZ(f, g, r, \theta, z)$. Recalling that the C_i's and the K_i's are functions of a only, we obtain: $H(a) = \frac{C_gC_f}{C_r^2} = \frac{C_gC_f}{C_zC_\theta} = \frac{C_g^2{C_f}^2}{C_\theta^2}$, and $K_f = K_g = K_r = 0$. These conditions are verified if $C_r^2 = C_zC_\theta$, $C_\theta = C_fC_gC_z$, which gives $C_r^2 = C_z^2C_fC_g \simeq C_z^2C_u$. We can then build an invariant $\eta(r,z,u)$ and then express KZ with variables η, θ.

Expression of an implicit invariant

Let us express KZ equation given by (7), initially written in r, z, θ variables in new transformed variables $\eta(r,z,u)$ and θ.

After large calculations (detailed in [6]), the new KZ is :

$$
\begin{aligned}
&-2gNrf[z]^2g_{\theta\theta} - rf[z]g_{\eta\eta}\eta_r^2 + g_\eta^2(-2gNrf[z]^4g_{\theta\theta}\eta_u^2\\
&-2f[z]^2(\eta_r(r\eta_{ru} - \eta_u) - r\eta_{rr}\eta_u)) - f[z]^3g_\eta^3\\
&\times(r\eta_{rr}\eta_u^2 + \eta_r\eta_u(-2r\eta_{ru} + \eta_u) + r\eta_r^2\eta_{uu}) + g_\theta^2\\
&\times(-2Nrf[z]^2 - 2gNrf[z]^4g_{\eta\eta}\eta_u^2 + g_\eta(2Nrf[z]^3\eta_u\\
&-2gNrf[z]^4\eta_{uu})) + rf[z]g_{\eta\theta}(\eta_z + g\eta_u f'[z])\\
&+g_\eta(-f[z]\eta_r - rf[z]\eta_{rr} + 4gNrf[z]^3g_{\theta\theta}\eta_u \qquad (8)\\
&-rf[z]^2g_{\eta\theta}\eta_u(\eta_z + g\eta_u f'[z]))+\\
&g_\theta(-4gNrf[z]^3g_{\eta\theta}\eta_u - rf[z]^3g_\eta^2(-\eta_{uu}\eta_z + \eta_u\eta_{zu})+\\
&rf'[z] + rf[z]^2g_{\eta\eta}\eta_u(\eta_z + g\eta_u f'[z])+\\
&g_\eta(4gNrf[z]^4g_{\eta\theta}\eta_u^2 - rf[z]\eta_u f'[z]\\
&+rf[z]^2(\eta_{zu} + g\eta_{uu}f'[z]))) = 0 \equiv KZ(\eta, \theta)
\end{aligned}
$$

According to the results given by Lie group theory, this equation only depends on η and θ. The 18 coefficients in front of the g_{ij} are necessarily dependent on functions F_i of η and θ. Independently of any particular initial or boundary condition, the ratio of $F_6(\eta, \theta) = -2N r\mathrm{f}[z]^2$ and $F_{15}(\eta, \theta) = r\mathrm{f}'[z]$ is necessarily a function of η and θ. This leads to the calculation of $f(z)$:

$$\frac{-2N r\mathrm{f}[z]^2}{r\mathrm{f}'[z]} = \frac{F_6(\eta, \theta)}{F_{15}(\eta, \theta)} = Cte,\ ie\ \ f(z) = \frac{1}{2NCz + D} \qquad (9)$$

where C and D are constants.

We can see that the Rudenko solution (3) is included in this general property. This result is interesting in the sense that, even if we cannot access to the calculation of the field $u(r, \theta, z)$, it is possible nevertheless to give its asymptotic behavior. This result is exactly the physical consequence of the diffraction effects of the acoustic field, negligible in $z = 0$ and leading to zero pressure when $z \to \infty$. This is one of the future trends of this method of reduction : elaborate some optimized condition of excitation after extracting judicious variables

Simplification of KZ equation

Let us rewrite KZ assuming $u(r, \theta, Z) = Zg(r, \theta, Z)$, with $Z = \frac{1}{2NCz+D}$. Initial KZ equation given in (1) becomes

$$2NZ[gg_{\theta\theta} + g_\theta g_\theta + Cg_\theta] + 2NCZ^2g_{\theta Z} + g_{rr} + g_r/r = 0. \quad (10)$$

Performing the same analysis, the invariance property (5) applied to (10) finally leads to $G[a] = \frac{C_g}{C_r^2} = \frac{C_g^2C_Z}{C_\theta^2} = \frac{C_gC_Z}{C_\theta}$ that is: $C_Z = \frac{C_g}{(C_r)^2} = G[a]$, $K_r = K_Z = K_g = 0$, $C_g = C_\theta$. With the use of condition $C_Z = \frac{C_g}{(C_r)^2}$, we can find a new judicious invariant expressed versus (Z, g, r). This invariant $\eta(r, Z, g)$ must verify Eq.(6):

$$(\alpha_r r)\frac{\partial\eta}{\partial r} + (\alpha_Z Z)\frac{\partial\eta}{\partial Z} + (\alpha_g g)\frac{\partial\eta}{\partial g} = 0, \qquad (11)$$

from which the solution $\eta(\sigma, \tau)$ is expressed as a function of variables $\sigma(g, r) = gr^{-\frac{\alpha_g}{\alpha_r}}$ and $\tau(r, Z) = r^{-\frac{\alpha_Z}{\alpha_r}}Z$.

Reduction of invariants *via* initial conditions

Let us suppose, with special initial condition, that the invariant $\eta(\sigma, \tau)$ is only dependent on τ, that is $\eta = r^\delta Z$, where δ is a constant to be determined. Starting from (10), KZ can be rewritten as

$$
\begin{aligned}
&r^{-2+\delta}Z\delta^2g_\eta + 2CNZg_\theta + 2NZg_\theta^2\\
&+r^{2(-1+\delta)}Z^2\delta^2g_{\eta\eta} + 2CNr^\delta Z^2g_{\eta\theta} + 2gNZg_{\theta\theta} = 0. \qquad (12)
\end{aligned}
$$

Coefficients in front of partial derivatives g_i and g_{ij} must be functions of η et θ only leading to $\delta = 2$ only. This is an important result which allows to simplify KZ assuming the initial hypotheses. Rewriting these two expressions $\eta = r^2Z = \frac{r^2}{2NCz+D}$ and $Z = \frac{1}{2NCz+D}$ as a function of r, θ, z, we confirm the existence of a solution of KZ equation written in the form

$$u(r, \theta, z) = \frac{1}{2NCz + D}g(\frac{r^2}{2NCz + D}, \theta), \qquad (13)$$

which is the form of the Rudenko solution (3) with a suitable identification of constants.

Pseudo-separation of the normalized time

As done in the previous section for the characterization of $f(z)$, it is interesting to find a function $h[\theta]$ such as the KZ solution gives $u(r, \theta, z) = h[\theta]g(r, \theta, z)$. This leads to $K_h = K_g = K_r = 0$, and $H[a] = \frac{C_gC_h}{C_r^2} = \frac{C_gC_h}{C_zC_\theta} = \frac{C_g^2C_h^2}{C_\theta^2}$; that is $C_r^2 = C_zC_\theta$, $C_\theta = C_zC_gC_h \simeq C_zC_u$. An invariant $\eta(u, \theta, z)$ expanded as a function of u, θ, z can also be found for a simplification of KZ equation. With the same reduction method, we can shows that $h_1[\theta] = \exp(K_1\theta)$, which is exactly a part of the Lapidus and Rudenko solution (3) with $K_1 = i$. Nevertheless, we can use other invariants which leads to an other possible solution: $h_2[\theta] = K_2\theta^2$ which can be seen as a secular solution. The existence of secular solution appears in acoustic phenomenon leading to chock waves, but generally these solutions are linear time dependent.

Conclusion

Symmetries of KZ equation using Lie group approach has been used for showing remarkable properties of KZ solutions: invariance properties confirm the existence of an implicit solution ; equation (9) confirms the pseudo-separation of z which is the consequence of the far field diffraction; the last result exposes the explicit normalized time dependence θ of oscillations of the near field, and exhibit the possibility of a secular behavior proportional to θ^2

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