Transform Methods for Horizontally Layered Isotropic and Anisotropic Media with Obstacles

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Anisotropic soil model with horizontal stratification

In a former paper[1] a model for orthotropic and anisotropic material parameters in horizontally stratified soils is presented. The homogenous solution for the different wave types in a layer is derived using Fourier transforms about time t and all orthogonal coordinates in space. Axis x and y are defined as coordinates in the horizontal plane and z heads towards depth. The characteristic equations defining the relation between frequency and wave numbers are derived using an equilibrium of stresses based on the transformed relations defining the relation between stress and displacements. In the transformed domain the derivative relations between displacements and strains are reduced to normal equations and therefore the final relations are derived using matrix algebra. The flexibility matrix **F** has to be inverted to derive the relation between stresses and strains before this matrix is multiplied by the matrix **D** of the strain displacement relations in the transformed domain. The singular values of the matrix for equilibrium of stresses are used to derive the vertical wave number k_z for any combination of angular frequency ω and horizontal wave numbers k_x and k_y. The eigenvectors of the matrix define the direction and type of the waves. These eigenvectors are used to set up the equilibrium of displacements at the interfaces of the layers. The equilibrium of stresses at the interfaces needs the stress eigenvectors. These vectors are derived by putting the singular values into the displacement stress relationship and the displacement relations and applying the displacement eigenvectors. Application of Dirac functions depending on the eigenvalues allows the definition of generalized functions for the waves in the fully transformed domain. The eigenvectors of displacements and stresses are multiplied by an unknown constant and a Dirac function depending on the difference of the wave number k_z and the eigenvalues $k_{z,i}$ belonging to the eigenvectors Ψ_i. This function is transformed back about the vertical axis. The results are exponential functions depending on the eigenvalues $k_{z,i}$, the imaginary unit and depth z scaled with the eigenvectors and unknown constants. The exponential functions are evaluated numerically at the depths of the interfaces. The resulting equations are used to substitute the unknown internal constants by the displacements reducing the number of unknowns and simplifying the global matrix analogous to the finite element method. Using a singular frequency and double back transform about the vertical axis and time, the direction of waves in a half space is derived. Waves that propagate into the undesired direction are eliminated. Using Sommerfeld´s radiation condition, only waves propagating into depth fulfill causality.

Green´s function

The model so far is based on the homogenous solution and only allows for loads at the interfaces. To derive a complete set of Green´s functions for a stratified medium, a unit Dirac stress load is needed in arbitrary depth z facing into one of the directions x, y or z. The results are Green´s functions of stresses and displacements for dipoles. Monopole solutions needed for the Burton Miller method are added using the same procedure. The load is applied in an arbitrary depth z inserting an additional interface inside the loaded layer. This method is simple, but needs a new solution of the global matrix for every depth of the load. A procedure is presented that maps the boundary conditions from the internal interface to the interface at the top of the loaded layer using the same model for all Green´s functions. Only the global load vector describing the loads at the interfaces has to be evaluated and backward substitution is needed to determine the unknown displacements at the interfaces. In the calculation of internal stresses and displacements for a selected load, an additional particular part has to be added within the loaded layer from the top down to the depth of the load. The transformation from a load position in the layer to the top of the layer leads to a load and a gap in the displacement that is converted into an additional load using the stiffness matrix of the element matrix of the layer.

Loads at an internal interface

The model is invariant with respect to shifts in the horizontal directions. Therefore it is assumed that the impact acts at the origin $x=0$ and $y=0$. In the vertical direction the impact is applied in depth z_p within a layer of height d. A unit impact load p – is assumed to act into one of the three directions x, y or z.

$$\begin{aligned} p(x,y,z,t) &= \delta(x)\delta(y)\delta(z-z_p)\delta(t) \\ \hat{p}(k_x,k_y,z,\omega) &= \delta(z-z_p) \qquad , \; z_p = d_p - d/2 \end{aligned} \qquad \text{eq. 1}$$

The Dirac function is Fourier transformed to unit one. Instead of one singular position the complete spectrum depending on two wave-numbers k_x and k_y and the angular frequency ω is needed. Transforming local effects like impacts seems to be a bad idea, but the complete spectrum is already needed in the homogenous case, and, if the particular impact responses can be added as an additional load vector, only a small number of additional calculations is needed. Advantages are the decoupling of the equations for every wave number and angular frequency giving small systems of equations and allowing for parallelization. Communication is only needed in the pre- and post process. However, only the homogenous solution that does not allow for a load within the layer is given in the basic formulation. If we insert an additional interface in the depth of the load, the load acts at the interface and only the homogenous solutions are needed.

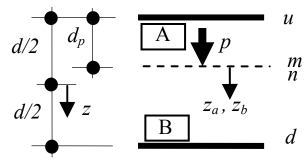

Fig.1: Additional interface and definition of parameters and stresses

To simplify the expressions, the vertical coordinates of the two layers start at the internal interface with the same global direction. In layer A the upper original interface is at the position $z_a=-d_p$. At the lower layer B the lower interface is at position $z_b=d-d_p$. With these assumptions the equations at the internal interface are simplified.

$$
\begin{aligned}
u:\hat{\mathbf{u}}_u &= \sum_{i=1}^{6} A_i \boldsymbol{\Psi}_i e^{-jk_{z,i}d_p} &,\quad \hat{\boldsymbol{\sigma}}_u &= \sum_{i=1}^{6} A_i \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_i e^{-jk_{z,i}d_p} \\
m:\hat{\mathbf{u}}_m &= \sum_{i=1}^{6} A_i \boldsymbol{\Psi}_i &,\quad \hat{\boldsymbol{\sigma}}_m &= \sum_{i=1}^{6} A_i \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_i \\
n:\hat{\mathbf{u}}_n &= \sum_{i=1}^{6} B_i \boldsymbol{\Psi}_i &,\quad \hat{\boldsymbol{\sigma}}_n &= \sum_{i=1}^{6} B_i \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_i \\
d:\hat{\mathbf{u}}_d &= \sum_{i=1}^{6} B_i \boldsymbol{\Psi}_i e^{jk_{z,i}(d-d_p)} &,\quad \hat{\boldsymbol{\sigma}}_d &= \sum_{i=1}^{6} B_i \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_i e^{-jk_{z,i}(d_p-d)}
\end{aligned}
\quad , \quad \text{eq. 2}
$$

The matrices **D** (strain displacement relationship, transformed differential matrix) and **F** (strain stress relationship, flexibility matrix) and the singular vectors $\boldsymbol{\Psi}_\mathbf{i}$ are identical in both layers because the layers consist of the same material. Only those three components of the six stresses which act on the horizontal interface are included. Compatibility of displacements at the interface and equivalence of stresses allows the reduction of the number of unknowns from 12 to 6 as it would have been without the additional interface. The equilibrium of stresses are non homogenous equations.

$$
\begin{bmatrix} \hat{\mathbf{u}}_m \\ \hat{\boldsymbol{\sigma}}_m \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{u}}_n \\ \hat{\boldsymbol{\sigma}}_n \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{p}} \end{bmatrix} \;, \hat{\mathbf{u}} = \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \\ \hat{u}_z \end{bmatrix} \hat{\boldsymbol{\sigma}} = \begin{bmatrix} \hat{\sigma}_{xz} \\ \hat{\sigma}_{yz} \\ \hat{\sigma}_{zz} \end{bmatrix} \qquad \text{eq. 3}
$$

Both divisions have identical material parameters. For this reason the matrices transforming the unknown scales in the upper and lower layer are identical.

$$
\begin{aligned}
&\underbrace{\begin{bmatrix} \boldsymbol{\Psi}_1 & \cdots & \boldsymbol{\Psi}_6 \\ \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_1 & \cdots & \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_6 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} A_1 \\ \vdots \\ A_6 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\Psi}_1 & \cdots & \boldsymbol{\Psi}_6 \\ \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_1 & \cdots & \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_6 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} B_1 \\ \vdots \\ B_6 \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{p}} \end{bmatrix}}_{\mathbf{f}} \\
&\mathbf{MA} = \mathbf{MB} + \mathbf{f} \\
&\mathbf{A} = \mathbf{B} + \mathbf{M}^{-1}\mathbf{f} = \mathbf{A}_{\text{hom}} + \mathbf{A}_{part} \Rightarrow \mathbf{A}_{\text{hom}} = \mathbf{B} \;,\; \mathbf{A}_{part} = \mathbf{M}^{-1}\mathbf{f}
\end{aligned}
\qquad \text{eq. 4}
$$

The unknowns of the homogenous part of the upper A_{hom} layer are identical to the unknowns B in the lower layer. Therefore the two layers are melting to one layer again. This effect simplifies the approach because the system equations are from now on independent from the load position.

$$
\begin{aligned}
\hat{\mathbf{u}}_u &= \hat{\mathbf{u}}_{u,\text{hom}} + \hat{\mathbf{u}}_{u,part} = \sum_{i=1}^{6} \Big\{ \underbrace{C_i}_{\text{hom}} + \underbrace{\left[\mathbf{M}^{-1}\mathbf{f}\right]_i e^{-jk_{z,i}z_p}}_{part} \Big\} \boldsymbol{\Psi}_i e^{-jk_{z,i}d/2} \\
\hat{\boldsymbol{\sigma}}_u &= \hat{\boldsymbol{\sigma}}_{u,\text{hom}} + \hat{\boldsymbol{\sigma}}_{u,part} = \sum_{i=1}^{6} \Big\{ \underbrace{C_i}_{\text{hom}} + \underbrace{\left[\mathbf{M}^{-1}\mathbf{f}\right]_i e^{-jk_{z,i}z_p}}_{part} \Big\} \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_i e^{-jk_{z,i}d/2} \\
\hat{\mathbf{u}}_d &= \hat{\mathbf{u}}_{d,\text{hom}} = \sum_{i=1}^{6} \Big\{ \underbrace{C_i}_{\text{hom}} \Big\} \boldsymbol{\Psi}_i e^{jk_{z,i}d/2} \\
\hat{\boldsymbol{\sigma}}_d &= \hat{\boldsymbol{\sigma}}_{d,\text{hom}} = \sum_{i=1}^{6} \Big\{ \underbrace{C_i}_{\text{hom}} \Big\} \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Psi}_i e^{jk_{z,i}d/2}
\end{aligned}
\qquad \text{eq. 5}
$$

The particular solution is calculated at the upper boundary. The indexed brackets $[]_i$ are used in Eq. 5 as a symbol for the extraction of the indexed component from the vector in the brackets. It becomes visible that stresses and displacements are not homogenous at the upper interface, projecting the internal load to that interface. Also, the particular parts of the equations belonging to the upper section of the original layer have to be added to the solution in the post process whenever stresses or displacements in the upper section are evaluated. The gap in the homogenous displacements at the upper interface caused by the non-zero vector side of the matrix equation is compensated by the particular part of the displacement function in the upper layer. The unknowns B_i are substituted by a set of unknowns C_i that belong to an origin of the vertical coordinate starting in the middle of the layer according to the definition for unloaded layers. For the numerical implementation, the parts in the brackets and outside the brackets are calculated separately and combined for different load cases and depths because the parts in the brackets depend only on the load case and the factors outside the bracket on the specific depth.

Using a vector **C** of the unknowns and a vector **P** of the particular parts of the equations, the homogenous and particular parts of the equation are separated one from each other.

$$
\begin{aligned}
&\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\Theta}_u \\ \boldsymbol{\Theta}_d \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_1 e^{-jk_{z,1}d/2} \ldots \boldsymbol{\Psi}_6 e^{-jk_{z,6}d/2} \\ \boldsymbol{\Psi}_1 e^{jk_{z,1}d/2} \ldots \boldsymbol{\Psi}_6 e^{jk_{z,6}d/2} \end{bmatrix},\; \Xi = \begin{bmatrix} \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Theta}_u \\ \mathbf{F}^{-1}\mathbf{D}\boldsymbol{\Theta}_d \end{bmatrix} \\
&\mathbf{P} = \begin{bmatrix} \mathbf{P}_u \\ \mathbf{0} \end{bmatrix}, \mathbf{P}_u = \left[\left[\mathbf{M}^{-1}\mathbf{f}\right]_1 e^{-jk_{z,1}z_p} \quad \cdots \quad \left[\mathbf{M}^{-1}\mathbf{f}\right]_6 e^{-jk_{z,6}z_p} \right]^T \\
&\qquad \hat{\boldsymbol{\sigma}}_{part} = \Xi\mathbf{P} \;,\; \hat{\boldsymbol{\sigma}}_{\text{hom}} = \Xi\boldsymbol{\Theta}^{-1}\hat{\mathbf{u}}_{\text{hom}}
\end{aligned}
\qquad \text{eq. 6}
$$

The homogenous equation is identical with the equation for an unloaded layer. The particular stresses define a global load vector. The displacements are weighted with the local stiffness matrix and the results are added to the load vector. This method needs some additional operations using the elemental stiffness matrix of the layer defining the stress displacement relations at the upper and lower boundary, but the degrees of freedom are reduced. Only the three homogenous displacements at the interface are given as unknowns instead of six coefficients in **C**. The post process for several loads remains nearly unchanged at the interface between unloaded layer 1 and loaded layer 2. Eq. 7 is derived.

$$
\begin{aligned}
&\hat{\mathbf{K}} = \Xi\boldsymbol{\Theta}^{-1} = \begin{bmatrix} \hat{\mathbf{K}}_{uu} & \hat{\mathbf{K}}_{ud} \\ \hat{\mathbf{K}}_{du} & \hat{\mathbf{K}}_{dd} \end{bmatrix} \\
&\begin{bmatrix} \hat{\mathbf{K}}_{2,uu} & \hat{\mathbf{K}}_{2,ud} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_{1,\text{hom},d} \\ \hat{\mathbf{u}}_{2,\text{hom},d} \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{K}}_{1,du} & \hat{\mathbf{K}}_{1,dd} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_{1,\text{hom},u} \\ \hat{\mathbf{u}}_{1,\text{hom},d} \end{bmatrix} = \\
&= \hat{\mathbf{K}}_{2,uu}\hat{\mathbf{u}}_{2,part,u} - \hat{\boldsymbol{\sigma}}_{2,part,u}
\end{aligned}
\qquad \text{eq. 7}
$$

Example, validation of the model, acknowledgement

Fig.2 presents the surface response of a stiff layer with 0.2 m depth on a soft subgrade. The load is an impact in vertical direction. The figures present the decay of the magnitude along the surface for an angular frequency ω=10000 rad/sec. No difference to the classical model with an additional interface at the load was detected. The project is supported by the Austrian Science Fund FWF Project P16224-NO7.

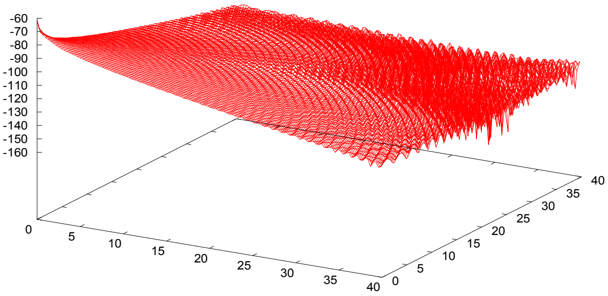

Fig.2: Top displacement of two layers on a half space with load at the internal interface (homogenous model)

[1] H. Waubke, P. Balasz: Verwendung der zeitlichen Rücktransformation zur Berücksichtigung der Kausalität in Spektren mehrdimensionaler Fourier Transformationen, DAGA 2003, No.1498