Propagation of Lamb waves in anisotropic rough plates : a perturbation method

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Introduction

Rough surfaces have been the subject of many studies involving the propagation of Rayleigh type waves [1-3]. For applications involving internal interfaces guided waves, guided waves such as Lamb waves are more useful. In this paper, the propagation of Lamb waves in an anisotropic plate with a randomly rough surface on one side, the other side being considered as the reference side, is studied. A 3D model is developed for an anisotropic plate in vacuum, characterized by its thickness d, its density ρ and its (6x6) elastic constant matrix $(c_{\alpha\beta})$. The boundary surface $x_3 = H(x_1, x_2) = -d/2 + h(x_1, x_2)$ has a weak variation $h(x_1, x_2)$ about the plane $x_3 = -d/2$ (see Fig. 1). The slopes $h'_1 = \partial h/\partial x_1$ and $h'_2 = \partial h/\partial x_2$ are also assumed to be small. A perturbation method is presented in order to express the dispersion equation of the rough plate as a sum of the dispersion equation of the plate with roughless surfaces and of a perturbation.

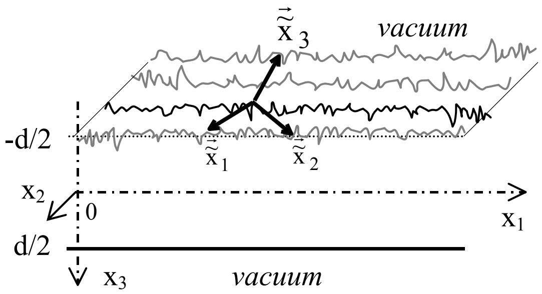

Figure 1: Geometry of the problem

Theoretical model

Change of basis

Note $\tilde\sigma_{ij}$ and σ_{kl} the coefficients of the stress tensor expressed respectively in the local basis $\tilde{\mathcal{B}} = (\tilde{\vec x}_1, \tilde{\vec x}_2, \tilde{\vec x}_3)$ (linked to each point $M(\vec x)$ of the surface $x_3 = H(x_1, x_2)$, $\tilde{\vec x}_3$ being the normal vector to the upper surface) and in the cartesian basis. $\tilde\sigma_{ij}$ and σ_{kl} are related by the tensor formula

$$\tilde\sigma_{ij} = a_{ik}\, a_{jl}\, \sigma_{kl} \quad , \tag{1}$$

where a_{ik} are the coefficients of the change-of-basis matrix for $\mathcal{B}$ to $\tilde{\mathcal{B}}$, which depend on h'_1 and h'_2. Using Eq. (1) permits to write the following matricial relation

$$\tilde\sigma(x_3) = J\, \sigma(x_3) \, , \tag{2}$$

where J is a (3x6) matrix, $\tilde\sigma(x_3) = (\tilde\sigma_{33}, \tilde\sigma_{23}, \tilde\sigma_{13})^T$ and $\sigma(x_3) = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})^T$ are respectively the (3x1) column vector made up of the three components of the stress vector linked to the normal vector $\tilde{\vec x}_3$ to the upper surface and the (6x1) column vector made up of the six components of the stress tensor, T denoting the transpose operation.

By introducing the slowness vector ${}^{(\eta)}\vec m$ of the wave (η) in the plate, the particular displacement vector can be written as

$$\vec u(\vec x; t) = \sum_{\eta=1}^{6} {}^{(\eta)}a\; {}^{(\eta)}\vec P\, e^{-i\omega\left({}^{(\eta)}\vec m\cdot\vec x - t\right)} \, , \tag{3}$$

where ${}^{(\eta)}a$ and ${}^{(\eta)}\vec P$ are respectively the displacement amplitude and the polarisation vector of the wave (η), and ω is the angular frequency of the waves.

The writing of Hooke's law [4] allows to express $\sigma(x_3)$ as a function of the (6x1) column vector $\mathcal{A}$ made up of the six displacement amplitudes ${}^{(\eta)}a$:

$$\sigma(x_3) = D\,\mathcal{H}(x_3)\,\mathcal{A} \, , \tag{4}$$

omitting the factor $-i\omega e^{-i\omega\left({}^{(\eta)}\vec m\cdot\vec x_1 + {}^{(\eta)}\vec m\cdot\vec x_2 - t\right)}$. The (6x6) matrix D only depends of the elastic constants $c_{\alpha\beta}$, of the slowness vector ${}^{(\eta)}\vec m$ and of the polarisation vector ${}^{(\eta)}\vec P$. The (6x6) matrix $\mathcal{H}(x_3)$ is a diagonal matrix

$$\mathcal{H}(x_3) = \mathrm{diag}\left(e^{-i\omega\, {}^{(\eta)}m_3\, x_3}\right) \, , \tag{5}$$

where ${}^{(\eta)}m_3$ is the projection on the x_3-axis of ${}^{(\eta)}\vec m$. Substituting Eq. (4) into Eq. (2) leads to

$$\tilde\sigma(x_3) = J\,D\,\mathcal{H}(x_3)\,\mathcal{A} \quad . \tag{6}$$

Second-order expansion

A second-order expansion of all the coefficients of the change-of-basis matrix for $\mathcal{B}$ to $\tilde{\mathcal{B}}$ permits to express the (3x6) matrix J as a linear combination of six matrices :

$$J \approx J_0 + h'_1 J_{x_1} + h'_2 J_{x_2} + h'_1 h'_2 J_{x_1 x_2} + h'^2_1 J_{x_1 x_1} + h'^2_2 J_{x_2 x_2} \tag{7}$$

Using Eq. (6), the stress vector $\tilde\sigma(x_3)$ is thus also expanded at a second-order in h'_1 and h'_2.

A second-order expansion in h of the exponential propagation factors of the diagonal matrix $\mathcal{H}(x_3)$ about $x_3 = -d/2$ enables to express the matrix $\mathcal{H}(x_3)$ as a linear combination of three matrices :

$$\mathcal{H}\left(-d/2 + h\right) \approx \mathcal{H}_0^- + h\,\mathcal{H}_1^- + h^2\,\mathcal{H}_2^- \, . \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6) leads to a second-order expansion of the stress vector $\tilde{\sigma}(x_3)$ at the upper surface. This expansion is the sum of eighteen matrices, the zero-order term being $J_0 D \mathcal{H}_0^-$.

Boundary conditions

As the plate is in vacuum, the stress vector linked to the normal of the interfaces has to be zero :

$$\tilde{\sigma}\left(-d/2+h\right)=0 \quad \text{at} \quad x_3=H(x_1,x_2)\ , \qquad (9)$$

and

$$\hat{\sigma}\left(d/2\right)=0 \quad \text{at} \quad x_3=d/2\ . \qquad (10)$$

$\hat{\sigma}(x_3)=(\sigma_{33},\sigma_{23},\sigma_{13})^T$ is the (3x1) column vector made up of the three components of the stress vector linked to the normal vector $\vec{x}_3$ to the lower surface :

$$\hat{\sigma}(x_3)=J_0 D \mathcal{H}(x_3)\mathcal{A}\ . \qquad (11)$$

The boundary conditions (9) and (10) lead to a 6-th order homogeneous system of equations :

$$M\mathcal{A}=0\ . \qquad (12)$$

M is a (6x6) matrix which is the sum of eighteen matrices and which can be expressed as follows

$$M \approx M_0+\delta M = M_0\left(I+M_0^{-1}\delta M\right)\ , \qquad (13)$$

where M_0^{-1} is the inverse of M_0 and I is the identity matrix. The matrix M_0 corresponds to the homogeneous system of equations, written for roughless surfaces and is given by :

$$M_0=\begin{bmatrix} J_0 D\mathcal{H}_0^- \\ J_0 D\mathcal{H}_0^+ \end{bmatrix} \quad \text{with} \quad \mathcal{H}_0^+=\mathcal{H}\left(x_3=d/2\right). \qquad (14)$$

The homogeneous system (12) has non zero solution only if the determinant $\det M$ of the matrix M is equal to zero,leading to the dispersion equation for Lamb modes, which can be written in the form

$$\det M = F(k_1,\omega)=0, \qquad (15)$$

where k_1 is the projection of the wave number vector on the x_1-axis, its real and imaginary parts being respectively noted k'_1 and k''_1. Eq. (15) can be expressed as follows

$$F\left(k_1,\omega\right)\approx F_0\left(k_1,\omega\right)+\delta F\left(k_1,\omega\right)=0\ . \qquad (16)$$

It can be noticed that the cancellation of the function $F_0(k_1,\omega)=\det M_0$ corresponds to the dispersion equation for Lamb modes in a plate with plane surfaces. In this case, for a given pulsation ω, the solution is real and is denoted k_0. Thus, it can be assumed that, for a given pulsation ω, the solution k_1 of Eq. (15) is of the form :

$$k_1=k_0+\delta k_1\ . \qquad (17)$$

The roughness of the stress-free boundary induces a small complex perturbation δk_1 of the solution of the dispersion relation, the real and imaginary parts of which are related respectively to the shift frequency and to the attenuation of the wave. Two mechanisms contribute to the decay of a Lamb mode: its decay into bulk elastic waves and its decay into other Lamb modes, with an energy transfer between modes.

Numerical and experimental results

Experimental and numerical studies have been done on an rough shot blasted glass plate. The plate is isotropic but the roughness needs a 3D description. The profile of the surface is described by its statistical properties : its mean value R_a and the mean square deviation of the surface from the flatness R_q. For a random profile $H(x_1,x_2)$, the Lamb wave is sensitive to a kind of "average" parameters ($\ll\ \gg$) of the surface, involved in the dispersion equation (15) or (16) :

$$\alpha_1=\ll h'_1\gg\ ,\ \alpha_2=\ll h'_2\gg\ ,\ \beta=\ll h(x_1,x_2)\gg\ ,$$

and $\gamma=\ll h^2(x_1,x_2)\gg$. These parameters depend on the Lamb mode and on the spatial wavelength Λ (given by the Power Spectrum Density of the surface $S(x_1,x_2)$) corresponding to the roughness profile. As a first approach, α_1, α_2, and γ can be identified to $\alpha_1 \cong \alpha_2 \cong 4R_a/\Lambda$ and $\gamma \cong R_q^2$.

The roughness has a weak effect on the phase velocities (related to k'_1). On the other hand, though the wave numbers k_1 are real when the plate is smooth (their imaginary part is zero), the wave numbers become complex when the surface is rough (see Fig. 2). As a consequence, the roughness has a influence mainly on the attenuation (related to k''_1).

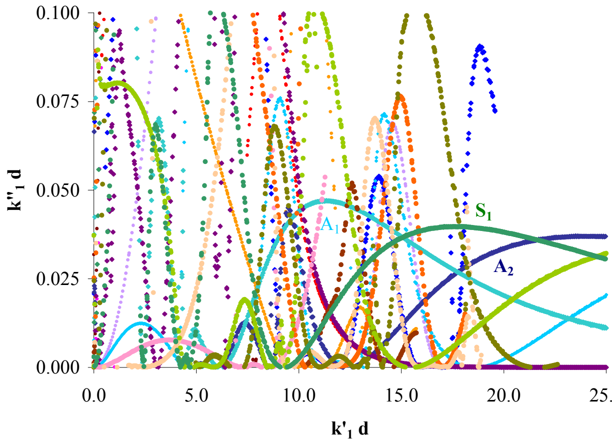

Figure 2: Dispersion curves in the complex plane of $k_1 d$ for a shot blasted glass plate ; $R_a = 23.3$ µm, $R_q = 29.8$ µm, $\Lambda = 0.546$ mm .

Experimental [5] and numerical results are in very good agreement for mode S_1 ($k''_1 d = 0.025$ in both cases, with an improvement with respect to a 2D model [6]) but are less good for other modes : the influence of the spatial wavelengths, through the PDS, has to be taken into account.

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