Numerical Calculation of Acoustic Scattering From Simple Structures Using the Multiparametric Gradient Method

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Introduction

The pressure field scattered from submerged bodies is of enormous interest in underwater acoustics. In some cases the three–dimensional Helmholtz differential equation can be reduced to the two–dimensional Kirchhoff–Helmholtz integral equation. But even this cannot usually be solved analytically for structures more complicated than simple spheres. Discretizising the continuous surface problem results in an N-dimensional system of linear equations that has to be solved. For our upcoming problems with $N \geq 10^5$ we are searching for an iterative solver. In this work we present a new gradient method and compare the results with the results of the known GMRES solver.

Theory

A scatterer S lies in an unbounded fluid and a source outside of S sends a monochromatic wave at an angular frequency $\omega = ck$ with c being the speed of sound and k being the wave number. Starting with the Helmholtz equation and using Gauss' integral theorem, the pressure $p(\vec{r})$ at a point $\vec{r}$ on the surface ∂S of a rigid scatterer can be formulated as a Fredholm equation of the second kind [6].

$$p(\vec{r}) \;=\; 2q(\vec{r}) + 2 \int\limits_{\vec{r'} \in \partial S'} K(\vec{r}, \vec{r'}) p(\vec{r'}) \; dS(\vec{r'}) \quad (1)$$

$$K(\vec{r}, \vec{r'}) \;=\; \frac{ik\mathrm{e}^{-ik|\vec{d}|}}{4\pi|\vec{d}|} \left[1 - \frac{i}{k|\vec{d}|}\right] \frac{\vec{n}(\vec{r'}) \cdot \vec{d}}{|\vec{d}|} \quad (2)$$

$\partial S'$ is equal to the surface ∂S but without the point $\vec{r}$ causing a singularity. $\vec{n}(\vec{r'})$ is the normal vector at point $\vec{r'}$ with $|\vec{n}| = 1$ pointing to the outside of S and $\vec{d} = \vec{r} - \vec{r'}$. $q(\vec{r})$ is the incident field. The unknown function $p(\vec{r})$ is projected onto a set of N known functions $p_n(\vec{r})$ using the projection:

$$p(\vec{r}) \;=\; \sum_{n=1}^{N} \alpha_n p_n(\vec{r}) \quad (3)$$

Instead of searching for the pressure itself we are looking for the coefficients α_n which give us the best approximation for $p(\vec{r})$ using linear combinations of the test functions $p_n(\vec{r})$. $p(\vec{r})$ in eq. 1 is replaced by the sum in eq. 3, the resulting equation is multiplied by the function $p_m(\vec{r})$. An extra integration over all $\vec{r} \in \partial S$ is carried out.

$$\Longrightarrow \sum_{n=1}^{N} A_{mn} \alpha_n = q_m \quad (4)$$

$$q_m = 2 \int\limits_{\vec{r} \in \partial S} p_m(\vec{r}) q(\vec{r}) dS \quad (5)$$

$$A_{mn} = \int\limits_{\partial S} p_m(\vec{r}) \left[p_n(\vec{r}) - 2 \int\limits_{\partial S'} K(\vec{r}, \vec{r'}) p_n(\vec{r'}) dS' \right] dS \quad (6)$$

Eq. 4 is a system of linear equations in N dimensions for a known vector $\vec{q}$ and a square matrix $\underline{A}$.

Iteration Scheme

Among others Kleinman et al. [1] used a formulation with a self adjoint operator, so we have also applied this. With $\underline{A}^H$ being the hermitian conjugate of $\underline{A}$, $\vec{Q} = \underline{A}^H \vec{q}$ and $\underline{B} = \underline{A}^H \underline{A}$ we have to solve

$$\underline{B} \vec{\alpha} = \vec{Q} \quad (7)$$

with the self adjoint operator $\underline{B}^H = \underline{B}$. The iteration scheme to solve eq. 7 follows the usual rules. After setting $i = 1$ and the start vector $\vec{\alpha}_{(1)}$ we

1. calculate the residual $\vec{R}_{(i)} := \vec{Q} - \underline{B}\vec{\alpha}_{(i)}$
2. derive a step vector $\vec{\Delta}_{(i)}$ from $\vec{R}_{(i)}$
3. do the step $\vec{\Delta}_{(i)}$ towards the exact solution $\vec{\alpha}_{(i+1)} := \vec{\alpha}_{(i)} + \vec{\Delta}_{(i)}$
4. update $i := i + 1$ and continue with step 1 if the desired error level is not yet reached.

Here we use

$$\vec{\Delta}_i = \sum_{l=1}^{L} \omega_{i,l} \underline{H}_{i,l} \vec{R}_{(i)} \quad (8)$$

with a set of L pre–conditioners $\underline{H}_l$ and their parameters ω_l. The index i at ω and $\underline{H}$ denotes that these quantities possibly have to be re–calculated at every iteration step. With a known set of $\underline{H}_{i,l}$ the potential function

$$\Psi_{(i)} = \frac{1}{2} \left| \vec{Q} - \underline{B}\vec{\alpha}_{(i)} \right|^2 \quad (9)$$

is minimized using variation of the paramerters $\omega_{i,l}$ by solving a system of L linear equations. As in most cases $L > 1$ is chosen and the gradient of a potential function is calculated, this iteration scheme is called **multi**parametric **gradient** method [4]. The well established GMRES [7] solver is the special case of this method with $L = i$. In this context the product $\underline{H}_{i,l}\vec{R}_{(i)}$ is called a search direction. In GMRES the system of equations that is used to minimize Ψ_i is growing with each iteration step. In this work we use a small system with $L = 2$ for every step and $\underline{H}_{i,1}\vec{R}_{(i)} = \vec{R}_{(i-1)}$ and $\underline{H}_{i,2}\vec{R}_{(i)} = \vec{R}_{(i-2)}$. This can be considered as a simplified GMRES solver in which we ignore all the directions that are older than two iteration steps. We made this explicit choice because we wanted to compare our results directly with the results of the FIELD program [2] in which a GMRES solver with a $\underline{A}^H$–preconditioner is implemented. This is equivalent to our formulation that is implemented in a program called FRED after the Fredholm equation.

Simple Structures

For all of the following calculations a plane wave was used as the incident pressure. The resulting pressure on the surface is used to calculate the target strength

$$TS\ [\text{dB}] \quad = \quad 20 \times \log_{10}\left(\frac{|\vec{r}|}{1m} \cdot \left|\frac{p_s(\vec{r})}{p_{in}(\vec{r})}\right|\right) \qquad (10)$$

where $p_s(\vec{r})$ is the scattered pressure at a point in the far field and $p_{in}(\vec{r})$ is the incident field at the point $\vec{r}$. We always started with $\vec{\alpha}_1$ being the plane wave approximation as in the FIELD program and a distance $|\vec{r}| = 100m$ for the calculation of the target strength.

Every integral over the surface was discretizised by

$$\int\limits_{\vec{r}\in\partial S} f(\vec{r})dS \simeq \sum_{n=1}^{N} f(\vec{c}_n)a_n \qquad (11)$$

and the following calculations were carried out with test functions $p_m(\vec{r})$ being constant on a facet and with every facet represented by its center of mass $\vec{c}$ and its area a (collocation). Therefore N is now the number of facets and also the dimension of the system of linear equations.

We used a numerical model of a rigid sphere with radius $\varrho = 1m$, represented by 2402 grid points and 4800 triangular facets of nearly equal size. Fig. 1 compares the three resulting target strengths at 1910 Hz ($k\varrho = 8$) using the programs FRED and FIELD and the analytic solution. Both iteration methods needed 6 iterations to reach an error as small as the discretization error, defined in [5] and fit perfectly to the analytical solution in the range from -150^o to 0^o. Only in the range from -180^o up to -150^o they differ slightly. FIELD calculates a target strength of 1 dB too high while the result of FRED is 1.5 dB too low.

A "cat's eye" is a more difficult structure to investigate. This is a massive sphere missing one octant, in this case, the octant with $x > 0$, $y > 0$ and $z < 0$ is missing. The incident plane wave has a wave vector parallel to $(-1, -1, 1)$ with $|\vec{k}\varrho| = 8$, which is the axis with the threefold symmetry and the wave is propagating directly into the "eye". The plane in which the target strength is calculated is the mirror plane of this "eye" with a normal vector $(-1, 1, 0)$. The structure again has a radius $\varrho = 1m$ and is now represented by 1922 grid points and 3840 triangular facets. The target strengths are shown in fig. 2 for the full 360^o circle so the asymmetry can be seen. FIELD needed 15 iterations to reach an error smaller than the discretization error. The results after 15 iterations and 45 iterations using FRED are also shown. Analytical solutions for this structure are not available. The result of the FIELD program cannot be reached with the same number of iterations, more are neccessary. This shows the need of more than two search directions when calculating more complex structures.

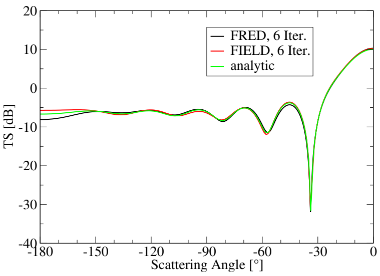

Figure 1: Comparison between different calculations of the target strength of a rigid sphere with $k\varrho = 8$. 0^o is the forward scattering, -180^o is the backward scattering.

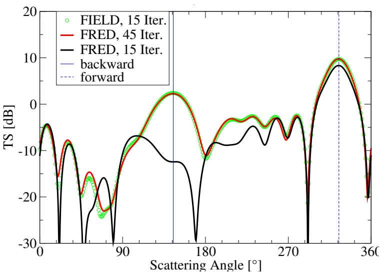

Figure 2: Comparison between different calculation of the target strength of a "cat's eye" with $k\varrho = 8$. The solid vertical line is the direction of backscattering, the dashed vertical line marks the forward scattering direction.

Conclusions

The presented multiparametric gradient method for the iterative solution of systems of linear equations gives the correct results for the scattered pressure field. The calculations using a sphere as a scatterer show that not all old search directions are necessary as they are used in the known GMRES solver. More difficult structures like the "cat's eye" show clearly the need of more than only two directions. The main advantage of the presented method is the freedom of choice for the preconditioners and for their number. With a more problem–orientated set of preconditioners than GMRES uses we are sure to significantly reduce the necessary number of iterations and therefore the calculation time.

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