Interpolation of complex frequency response curves

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Introduction

The problem at hand is best introduced with the help of an example, namely the problem of interpolating two functions as sketched in figure 1. Say, given two functions, f1 and f2, we want to create a third function f3, somewhere between f1 and f2 depending on the parameter p = 1 ... 0.

$$f_3 = p \cdot f_2 + (1 - p) \cdot f_1 \tag{1}$$

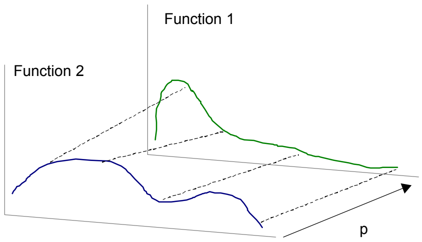

Figure 1: Interpolating two functions

Complex Interpolation

The interpolation formula (1) yields unexpected results if the start and end functions are complex valued. A practical example is an acoustic measurement of a loudspeaker on a turntable in order to measure the directivity. In this measurement an impulse response is taken for each angular step, say each 10 degrees.

Figure 2 demonstrates graphically as an example, how directivity data can be displayed. The contour belongs to a software module, which not only displays data in a variety of ways but mainly is also the container of the measured data. The module stores the complex frequency responses in a matrix for each turntable position.

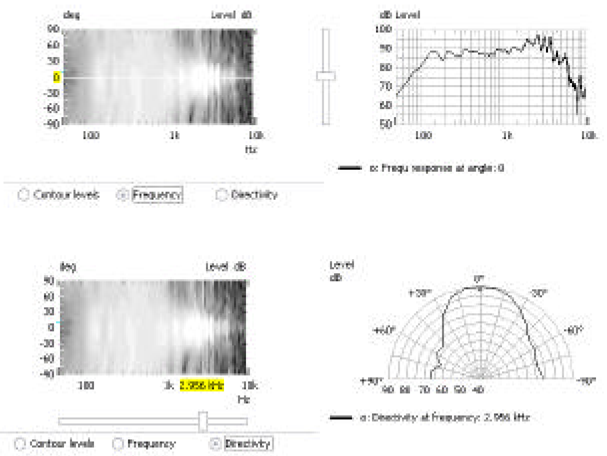

Figure 2: Contours: SPL of directivity measurement. x-axis: log-frequency, y-axis: linear radiation angles, z-axis: SPL in dB. Graphs: Mapping of a frequency response and a directivity polar plot

The aim is now to provide a function, which extracts data from this matrix and displays either a single frequency response at a certain angle, or, a single directivity polar plot at a certain frequency as demonstrated in figure 2. Both parameters, the angle and the frequency, should be arbitrary, i.e. also values in-between the measurement points should be allowed. Further, the total response, i.e. the complex valued data stream, should be mapped and stored in the module of the frequency or the polar plot, respectively. In this way further complex-valued processing can be performed on the mapped data, such as a Fourier transformation for example.

The problem is then to obtain a response in-between the angular measurement steps. If formula (1) is applied to the impulse response h1(t) at 0° and h2(t) at 10° then the interpolation rule is

$$h_3(t) = p \cdot h_2(t) + (1 - p) \cdot h_1(t) \tag{2a}$$

Or, in spectral form, after a time-frequency Fourier transformation:

$$H_3(j\omega) = p \cdot H_2(j\omega) + (1 - p) \cdot H_1(j\omega) \tag{2b}$$

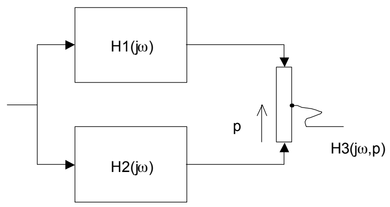

Figure 3: Mixing interpolation between H1 and H2

Graphically this mixing algorithm can be sketched as displayed in figure 3. The interpolated response is a mix of response H1 and H2 according the parameter p.

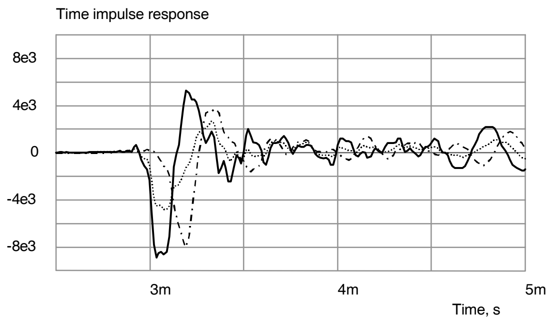

Figure 4: Mixing interpolation between systems h1(t) and h2(t). Solid: h3 at 0° (h3 = h1). Dotted: h3 at 5° (interpolated). Dashed: h3 at 10° (h3 = h2)

The time response version of the data is displayed in figure 4. The solid and dashed curves are the response at 0° and 10° as measured. The dotted curve interpolates according to the mixing rule, equations (2a). Because we have selected 5°, which is halve way through, the dotted curve shows the mean value of both responses (p = 1/2).

The frequency response version of the data is displayed in figure 5. The solid and dashed curves are again the response at 0° and 10° as measured. The dotted curve interpolates according to the mixing rule, equations (2b). However, in the frequency domain the result, although mathematically correct, does not yield the expected interpolation. We would like to have the interpolated amplitude curve (dotted) to be located somewhere in-between the solid and dashed curves.

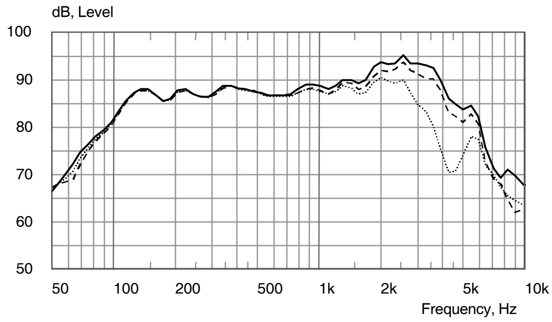

Figure 5: Mixing interpolation between H1(jω) and H2(jω), level in dB of complex frequency response, 3rd oct. smoothed. Solid: H3 at 0° (H3 = H1). Dotted: H3 at 5° (interpolated). Dashed: H3 at 10° (H3 = H2)

The question is, what went wrong, and then, which way to go for a better interpolation strategy, which yields reasonable interpolations, both in frequency and time domain.

Let us first have a closer look to the mixing formula 2b, which can be written in polar form:

$$H_3(j\omega) = p\cdot\left|H_2(j\omega)\right|\cdot e^{j\phi_2(j\omega)} + (1-p)\cdot\left|H_1(j\omega)\right|\cdot e^{j\phi_1(j\omega)} \quad (3)$$

The squared amplitude of H3 is then, after some manipulations:

$$\left|H_3(j\omega)\right|^2 = p^2\cdot\left|H_2(j\omega)\right|^2 + (1-p)^2\cdot\left|H_1(j\omega)\right|^2 + 2\cdot p\cdot(1-p)\cdot\left|H_2(j\omega)\right|\cdot\left|H_1(j\omega)\right|\cdot\cos(\phi_2(j\omega)-\phi_1(j\omega))) \quad (4)$$

Here, the main point to note is that no interpolation parameter, p occurs in the phase difference of the correlation term. That means, first, there is an interference phenomenon and, second, the amount of interference is fixed by the phase difference of H1 and H2. The dip of the interpolated curve in figure 5 (dotted) at high frequencies is hence caused by interference. If we imagine the extreme case, where H1 and H2 are identical but phase inverted, then formula (4) would yield identical zero for H3 for p = 1/2. This would be fine for the interaction of waves but misleading for the purpose of morphing one function into another one.

The "morphing" approach

An alternative way interpolating complex functions starts in the frequency plane:

$$H_3(j\omega) = \left[p\cdot\left|H_2(j\omega)\right| + (1-p)\cdot\left|H_1(j\omega)\right|\right]\cdot e^{j\cdot\left[p\phi_2(j\omega) + (1-p)\phi_1(j\omega)\right]} \quad (5)$$

Equation (5) interpolates the amplitude and phase separately. Immediately it is clear that the squared amplitude yields the expected interpolation behaviour:

$$\left|H_3(j\omega)\right|^2 = p^2\cdot\left|H_2(j\omega)\right|^2 + (1-p)^2\cdot\left|H_1(j\omega)\right|^2 \quad (6)$$

There is no phase dependence of the amplitude of H3 and, hence, no interference phenomenon. The result can be seen in figure 6. The amplitude (or level in this case) behaves as expected by smoothly morphing from one response to the other. The same is true for all the other components of the complex response, such as the real- and imaginary parts or the phase response.

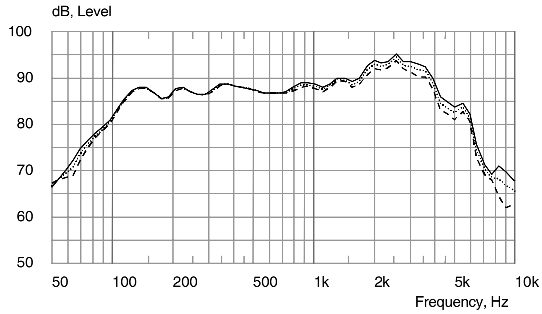

Figure 6: Morphing interpolation between systems H1(jω) and H2(jω), Level in dB of the complex frequency response, 3rd oct smoothed. Solid: H3 at 0° (H3 = H1)- Dotted: H3 at 5° (interpolated). Dashed: H3 at 10° (H3 = H2)

Most interesting however is the effect on the time response, h3(t), i.e. the inverse Fourier transform of H3(jω) as demonstrated in figure 7:

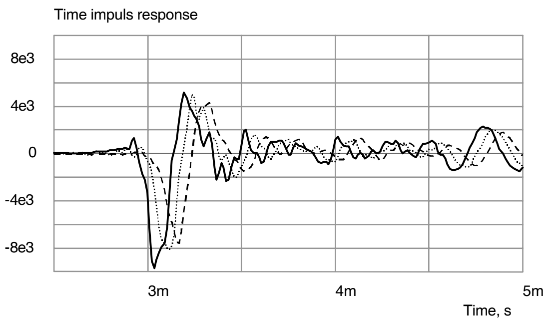

Figure 7: Morphing interpolation between systems h1(t) and h2(t), Solid: h3 at 0° (h3 = h1). Dotted: h3 at 5° (interpolated). Dashed: h3 at 10° (h3 = h2)

The dotted curve in figure 7 includes the time delay in order to morph response h1 into h2. This result is the one we would expect when interpolating response functions.

In polar coordinates a response function can be written

$$H(j\omega,t) = \left|H(j\omega)\right|\cdot e^{j\cdot(\phi(j\omega)+\omega\cdot t)} \quad (7)$$

The function exp(jΦ) is an infinity-to-one mapping because exp(jΦ) = exp(jΦ + j2πn). Hence, a continuous phase maps onto a circular repeating function with period 2π. Because exp(jΦ) is repeating itself, any interpolation in the exp(jΦ)-plane would need to count the cycles of revolution. Therefore, the interpolation is easier to perform in the phase plane, which is continuous.

The latter statement is may be "easier" from the theoretical point of view, but in practice we usually do not have access to a continuous phase function on which to perform the interpolation. In most cases the phase function is calculated from the real and imaginary parts by using the principal value of the arcus-tangens function, which is multi-valued.

However, there is a way to numerically unwrap the phase response into a continuous function. It can be shown that the continuous phase is:

$$\phi_c(j\omega) = Im\left\{\int_0^{\omega}\frac{1}{H(j\omega')}\cdot\frac{\partial H(j\omega')}{\partial\omega'}d\omega'\right\} + \phi_0 \quad (8)$$

ϕ_0 is an integration constant, which can be found by comparing the continuous phase with the principal phase.