Analytical solution for multi-modal acoustic propagation in circular ducts with Poiseuille flow

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Introduction

An analytical solution for the propagation of sound in circular ducts in the presence of laminar mean flow is derived. This solution, using Kummer's formalism, generalizes previous results found by Gogate and Munjal in the particular case of axi-symmetric modes [1]. In this short communication, the propagation equation is first derived. Effect of laminar shear flow is then discussed.

Governing equations

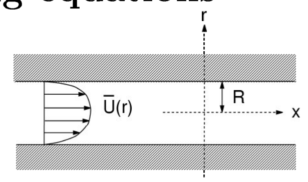

Figure 1: Configuration of the problem

We consider sound propagation in a circular duct with a laminar mean flow over the x-axis (see figure (1)). In such configuration, if the effect of viscosity is neglected, if it is assumed that the thermal conductivity of the fluid is negligible so that the entropy preturbations of the system can be taken to be zero and considering harmonic waves (acoustic pressure proportional to $P(r) \cdot \exp[\mathrm{i}(\gamma x - \omega t + m\theta)]$, γ being the propagation constant, ω the pulsation of wave and m the circumferential order of propagating mode), the combination of continuity equation and Navier-Stokes equation leads to the Pridmore-Brown equation (1)

$$\frac{\partial^2 P}{\partial \xi^2} - \left(\frac{1}{\xi} + \frac{\pm 4\mathcal{M}_0\Gamma\xi}{1 \mp \mathcal{M}_0\Gamma(1-\xi^2)}\frac{\partial \mathcal{M}}{\partial \xi}\right)\frac{\partial P}{\partial \xi} + \Omega^2\left((1 \mp \mathcal{M}_0\Gamma(1-\xi^2))^2 - \Gamma^2 - \frac{m^2}{\Omega^2\xi^2}\right)P = 0, \quad (1)$$

with the non-dimensional co-ordinate $\xi = r/R$, $\mathcal{M}(\xi) = \mathcal{M}_0(1-\xi^2)$ that describes the profile of mean flow, $\mathcal{M}_0$ being the centreline Mach number, and non-dimensional parameters $\Gamma = \gamma c_0/\omega$, $\Omega = \omega R/c_0$. Here and henceforth, upper/lower signs are to be taken for the downstream/upstream propagation respectively. Then, in the case of low Mach number, using $|\mathcal{M}_0\Gamma|^2 \ll 1$, the coefficients of the differential equation (1) can be developed at first order. This leads to the propagation equation

$$\frac{\partial^2 P}{\partial \xi^2} + \left(\frac{1-a\xi^2}{\xi}\right)\frac{\partial P}{\partial \xi} + \left(b + c\xi^2 - \frac{m^2}{\xi^2}\right)P = 0, \quad (2)$$

with $a = \pm 4\mathcal{M}_0\Gamma$, $b = \Omega^2(1 \mp 2\mathcal{M}_0\Gamma - \Gamma^2)$, and $c = \pm 2\mathcal{M}_0\Gamma\Omega^2$. This equation admits an analytical solution which is, introducing $\alpha = \sqrt{(a^2-4c)}$,

$$P(\xi) = \left(\frac{\alpha}{2}\right)^{\frac{m+1}{2}} \mathrm{e}^{\left(\frac{a-\alpha}{4}\xi^2\right)}\, \xi^m\, \mathbf{K}\left(A, m+1, \frac{\alpha}{2}\xi^2\right), \quad (3)$$

with $A = \frac{(m+1)\alpha-(a+b)}{2\alpha}$, where $\mathbf{K}$ is the Kummer's function of first kind. It can be shown that this solution converges on $\mathbf{J_m}(\sqrt{b}\xi)$ (the well-known no flow solution [3]) when the Mach number converges on zero. In the same way, it can also be shown [6] that the analytical solution (3) converges on uniform flow solution in the case of uniform flow.

Dispersion equation

Writing the boundary conditions allows to determine dispersion equation. Taking conditions of hard wall, that is

$$[v(\xi)]_{\xi=1} = \frac{\mathrm{i}}{\omega\rho_0 R}\left[\frac{\partial P(\xi)}{\partial \xi}\right]_{\xi=0} = 0. \quad (4)$$

leads to the following dispersion equation in Γ :

$$\frac{\alpha A}{m+1}\mathbf{K}\left(A+1, m+2, \frac{\alpha}{2}\right) + \frac{a-\alpha+2m}{2}\mathbf{K}\left(A, m+1, \frac{\alpha}{2}\right) = 0 = F(\Gamma). \quad (5)$$

Propagation constant Γ can be computed from this equation. For a given set of parameters ($\mathcal{M}_0$, Ω, and m) this equation admits an infinity of solutions. The real solutions correspond to propagative modes whereas the complex solutions correspond to evanescent modes.

Effect of laminar shear flow

The influence of laminar shear flow is discussed in this section. In order to obtain the propagation constant in the presence of a laminar shear flow, the function $F(\Gamma)$ is plotted numerically for a given set of parameters $\mathcal{M}_0$, Ω, and m. For instance, after computation of the propagative constants, profiles of mode (1,0) against Mach number (between 0 and 0.03) for two frequencies $\Omega = 4$ and $\Omega = 10$ are plotted in figure 2.

The no-flow solution is represented by a bold curve. The curve corresponding to the higher Mach number (0.3) is

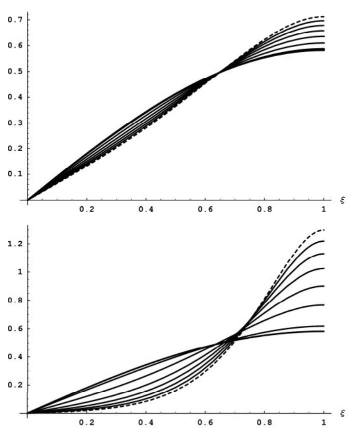

Figure 2: Profile of mode (1,0), at $\Omega = 4$ (upper case) and $\Omega = 10$ (lower case) for Mach number (0–0.3)

represented by a dashed line. The profiles have been normalized with respect to the no flow profile integrated over the section. These curves show that deformation of profile increases with increasing Mach number and increasing frequency, as expected by some authors [4, 5]. The deformation of profile is due to refraction effect. As expected, we also found that acoustic waves tends to be refracted towards the wall in downstream propagation whereas it tends to be refracted towards the core in upstream propagation. The influence of shear on propagation can also be shown with dispersion curves (Γ against Ω). Dispersion curves for propagating modes (m,n) are plotted on figure 3 at 0.2 Mach number. Upstream values were set into the negative for easier comparison and Mach number have been transformed into a Mach number associated with volume flow according to $\mathcal{M}_v = (1/S)\int_S \mathcal{M}(\xi)\mathrm{d}s = C\mathcal{M}_0$, for ease of comparison of the present results with previous works.

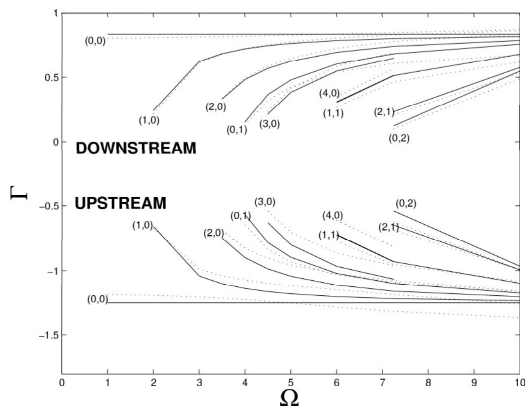

Figure 3: Dispersion curves at Mach number 0.2 for modes (m,n). Whole line : uniform flow, dotted line : laminar flow.

The difference between uniform flow and laminar flow is due to refraction effect. It increases with increasing frequency and is more important in upstream propagation than in downstream one. This present results are in agreement with numerical computations conducted by Bihhadi and Gervais [4]. In order to see influence of shear on propagative constant, figure 4 displays the difference between values of uniform flow propagative constant and laminar shear flow propagative constant.

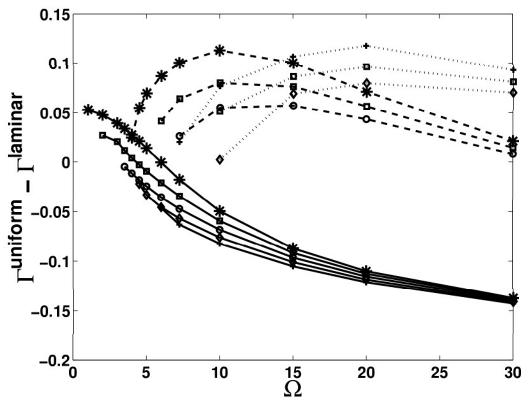

Figure 4: Difference between uniform flow and laminar flow constant against frequency ($\mathcal{M} = 0.3$). Whole line : 0 order radial mode, dashed line : 1 order radial mode, dotted line : 2 order radial mode.

Several behaviors are noticeable. For purely circumferential modes, (m,0), curves decrease with increasing frequency. Moreover, they seem to converge on the same asymptote. On the others hands, curves for radial modes ($n > 0$), present a bump before converging also to an asymptote. There are no similar results in the literature for comparison with the present ones as previous works did not consider the evolution of the propagation constant for non axi-symmetric modes. Even if the physical reason such behavior is left unexplained, our study shows that shear flow depends on the radial order of the propagating mode.

References

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