Une méthode semi-analytique adaptée pour l'étude du comportement acoustique des milieux poroélastiques hétérogènes

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A semi-analytical finite element method for studying acoustical behavior of heterogeneous poroelastic media is presented. Particularly we are interested in studying unidirectionally heterogeneous (multilayer or functionally graded) waveguide coupled with fluids. Biot's theory for anisotropic poroelastic medium is used. By transforming the wave equation into the frequency and wavenumber domain (which corresponds to homogeneous direction), one-dimensional finite element formulations are derived to numerically solve the equation in the heterogeneous direction. Numerical examples show that the proposed method is efficient for estimating the the reflection and transmission coefficients of porelastic plates.

1 Introduction

Modeling of wave propagation in porous waveguide has received much of attention in the past. It is motivated by characterization and optimization problems of different materials such as sound absorbing materials, industrial foams, biological materials (such as bone or wood), concrete, sandstone, etc. In many case of these applications, the macroscopic mechanical property of these waveguides is relatively homogeneous along longitudinal direction but inhomogeneous (with functionally graded or layered profile) in the cross section.

In this paper, the ultrasonic wave propagation in fluid saturated anisotropic waveguide is studied. By assuming that the wavelengths are larger than the average pore size and the porous medium is completely saturated by a fluid, the Biot's theory, which has been widely employed in many applications, is used here.

In order to study the wave propagation problem in functionally-graded/layered waveguides in the frequency domain, analytical methods, such as direct stiffness matrix method [2, 5], are usually used. Alternatively, for considering waveguides with geometrical and mechanical properties which are constant only along one or two directions, the Hybrid Numerical Method (HNM, see *e.g.* [3]), alternatively called Semi-Analytical Finite Element method (SAFE, see *e.g.* [1, 4]), have been employed. The key point of this method consists in using a hybrid algorithm which begins by employing the Fourier transform (with respect to time and to the longitudinal direction of the waveguide) to transform problem into the frequency-wavenumber domain. Then, the wave equations in the spectral domain governed in cross-section (or even a 1D domain in the case of infinite plates or axisymmetric wave-guides), which may actually have inhomogeneous material properties, can be easily handledusing the finite element method [4].

In this paper, we present a procedure for computing the reflection and transmission coefficients of anisotropic poroelastic plate by using SAFE method.

2 Governing equations

2.1 Problem description

Let $\mathbf{R}(\mathrm{O};\mathbf{e}_1,\mathbf{e}_2)$ be the reference Cartesian frame where O is the origin and $(\mathbf{e}_1,\mathbf{e}_2)$ is the orthonormal basis for the bi-dimensional space. The coordinates of a point $\boldsymbol{x}\in\mathbf{R}$ are denoted by (x_1,x_2) and the time is denoted by t As shown in Figure 1, we consider a poroelastic layer with thickness h, which occupies the unbounded domain Ω^b in $\mathbf{e}_1$ direction, is surrounded by two fluid half-spaces Ω_1^f and Ω_2^f. The interfaces between the poroelastic layer Ω^b and the fluid domains Ω_1^f and Ω_2^f are assumed to be flat and denoted by Γ_1^{bf} and Γ_2^{bf}, respectively.

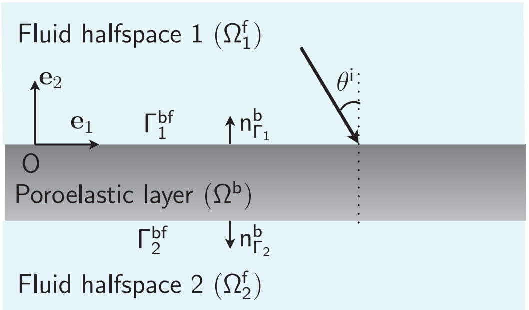

Figure 1: Geometry description

The surrounding fluids in Ω_1^f and Ω_2^f are assumed to be homogeneous and inviscid. The layer Ω^b is assumed to be a fully saturated and transversely isotropic poroelastic medium. We also assume that the physical properties of the porous layer is homogeneous along its longitudinal direction ($\mathbf{e}_2$) but may have an inhomogeneous profile in its depth direction ($\mathbf{e}_1$). Despite the fluid viscosity is neglected in the surrounding fluid domains, it is taken into account in the porous plate pores.

A plane and harmonic wave with an angular frequency ω, propagating in the upper fluid domain, is incident under angle θ_I to the interface Γ_1^{bf}. To determine the reflection and transmission coefficients of the poroelastic layer, we assume a time-dependence $e^{-\mathrm{i}\omega t}$ ($\mathrm{i}=\sqrt{-1}$) for all movement quantities $Y(\boldsymbol{x},t)$, *i.e* $Y(\boldsymbol{x},t)=y(\boldsymbol{x},\omega)e^{-\mathrm{i}\omega t}$. In the follows, the term ω in $y(\boldsymbol{x},\omega)$ will be omitted for simplification purposes.

2.2 Equations for wave propagation in the fluids (Ω_1^f and Ω_2^f)

We denote by ρ_1, ρ_2 the mass densities and c_1, c_2 the wave celerities of the fluids in Ω_1^f and Ω_2^f, respectively. In these domains, the Helmholtz and Euler equations read

$$-\frac{\omega^2}{c_n^2}p^{(n)}-p^{(n)}_{,jj}=0, \tag{1}$$

$$-\mathrm{i}\omega u^{(n)}+\frac{1}{\rho_n}\,p^{(n)}_{,j}=0, \tag{2}$$

where $p^{(n)}$ and $u_j^{(n)}$ denote the pressure and components of displacement vector of the fluid in Ω_n^f $(n=1,2)$.

Let us consider an incident plane acoustic wave p_I propagating with a pulsation ω in the upper fluid domain Ω_1^f and arriving to the interface Γ_1^{bf} from an angle θ_I. Then the total pressure in Ω_1^f may be expressed by: $p^{(1)} = p_I + p_R$ where p_R is the reflection plane wave field. By denoting $k_0 = \frac{\omega}{c_1}$ the wavenumber of p_I in in Ω_1^f, the pressure field p_I should satisfy (1) and is expressed by $p_I = P_I e^{\mathrm{i}(k_1 x_1 - k_2^{(1)} x_2)}$, where P_I denotes the wave amplitude; $k_1 = k_0 \sin\theta_I$ and $k_2^{(1)} = k_0 \cos\theta_I$. As a consequence, $p_R = P_R e^{\mathrm{i}(k_1 x_1 + k_2^{(1)} x_2)}$ where P_R is the amplitude of the reflected wave.

Similarly, the solution of plane the transmission wave in Ω_2^f reads:

$$p^{(2)} = P_T e^{\mathrm{i}(k_1 x_1 + k_2^{(2)}(x_2 + h))}, \tag{3}$$

where P_T is the amplitude of the transmitted wave and $k_2^{(2)} = \sqrt{\frac{\omega^2}{c_2^2} - k_1^2}$.

2.3 Wave propagation in the anisotropic poroelastic layer

The constitutive equations for an anisotropic poroelastic material are given by

$$\sigma_{jk} = C_{jklm}\epsilon_{lm} - \alpha_{jk}\, p\,, \tag{4}$$

$$-\frac{1}{M}\, p = w_{j,j} + \alpha_{jk}\epsilon_{jk}\,, \tag{5}$$

where σ_{jk} denote the components of the total stress tensor; ϵ_{jk} denote the components of the strain tensor: $\epsilon_{jk} = \frac{1}{2}(u_{j,k} + u_{k,j})$ with u_j are components the solid skeleton's displacement vector; w_j is the fluid/solid relative displacement weighted by the porosity: $w_j = \phi(u_j - u_j^f)$ with u_j^f denote the fluid displacements and ϕ denotes the porosity; α_{jk} are the Biot effective coefficients and M is the Biot's modulus.

Neglecting the body forces (other than inertia), the equations describing the linear poroelastic wave propagation in the frequency domain read:

$$\sigma_{jk,k} = -\omega^2 \rho\, u_j - \omega^2 \rho_f\, w_j\,, \tag{6}$$

$$-p_{,j} = -\omega^2 \rho_f\, u_j - \omega^2 \tilde{a}_{jk} w_k\,, \tag{7}$$

where $\rho = \phi\,\rho_f + (1-\phi)\,\rho_s$ is the mixture density, with ρ_s and ρ_f are the solid and fluid densities, respectively; $\tilde{a}_{ij}$ are component of a frequency dependent visco-dynamic tensor which depends on the permeability and tortuosity of the medium. For a transversely isotropic poroelastic material, $\tilde{\boldsymbol{a}}$ is a diagonal tensor of which $\tilde{a}_{jj},\ j = 1, 2$ are the dynamic tortuosity in $\mathbf{e}_1$ and $\mathbf{e}_2$ directions and may be estimated by

$$\tilde{a}_j(\omega) = \frac{\rho_f}{\phi}\left(a_j^\infty + \frac{\mathrm{i}\phi\eta F_j(\omega)}{\omega\rho_f\kappa_j}\right) \tag{8}$$

where a_j^∞ is the static tortuosity, η is the viscosity of the interstitial fluid, κ_j is the intrinsic permeability in direction j; $F_j(\omega)$ are the correction factors which are introduced to take into account the viscous resistance of the fluid flow at high frequencies. We recall that all mechanical characteristics above are x_2-dependent functions.

In the sequel, we rewrite the equations below in matrix form. Let us then use Voigt's notation which expresses the symmetric second-order tensors as vectors, so the stress is denoted $\boldsymbol{s} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$, the strain by $\boldsymbol{e} = \{\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12}\}^T$, the Biot effective coefficients by $\check{\boldsymbol{\alpha}} = \{\alpha_{11}, \alpha_{22}, \alpha_{12}\}^T$ where the superscript $\star^T$ designates the transpose operator. We also introduce an operator $\mathbf{L}$ which takes the form: $\mathbf{L} = \mathbf{L}_1\partial_1 + \mathbf{L}_2\partial_2$ with:

$$\mathbf{L}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{L}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{9}$$

where ∂_1 and ∂_2 denote the partial differentiation operators with respect to x_1 and x_2, respectively. Using these notations, the balance equations of linear momentum (6)-(7) may be rewritten as:

$$-\omega^2\rho\ \boldsymbol{u} - \omega^2\rho_f \boldsymbol{w} - \mathbf{L}^T\boldsymbol{s} = \mathbf{0}, \tag{10}$$

$$-\omega^2\rho_f \boldsymbol{u} - \omega^2\tilde{\boldsymbol{a}}\ \boldsymbol{w} + \mathbf{L}^T\boldsymbol{m}p = \mathbf{0}, \tag{11}$$

where $\boldsymbol{m} = \{1, 1, 0\}^T$. The constitutive equations (4)-(5) read:

$$\boldsymbol{s} = \mathbf{C}\ \boldsymbol{e} - \check{\boldsymbol{\alpha}}\ p, \tag{12}$$

$$p = -M\left(\boldsymbol{m}^T\,\mathbf{L}\boldsymbol{w} + \check{\boldsymbol{\alpha}}^T\mathbf{L}\boldsymbol{u}\right), \tag{13}$$

where $\mathbf{C}$ is the drained elastic tensor using Voigt's notation:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}. \tag{14}$$

By noting that $\boldsymbol{e} = \mathbf{L}\boldsymbol{u}$ and by substituting Eq. (13) into Eq. (12), the constitutive equations (12)-(13) may be written by

$$\boldsymbol{s} = \mathbf{C}_u\ \mathbf{L}\ \boldsymbol{u} + \mathbf{C}_\alpha\ \mathbf{L}\ \boldsymbol{w}, \tag{15}$$

$$\boldsymbol{m}p = -\left(\mathbf{C}_M\mathbf{L}\boldsymbol{w} + \mathbf{C}_\alpha^T\mathbf{L}\boldsymbol{u}\right), \tag{16}$$

where $\mathbf{C}_u = \mathbf{C} + M\check{\boldsymbol{\alpha}}\check{\boldsymbol{\alpha}}^T$, $\mathbf{C}_\alpha = M\check{\boldsymbol{\alpha}}\boldsymbol{m}^T$, $\mathbf{C}_M = M\boldsymbol{m}\boldsymbol{m}^T$. The tensor $\mathbf{C}_u$ is known as the undrained elasticity tensor which represents the rigidity of an equivalent elastic medium in which the relative movement between the interstitial fluid and solid skeleton is vanished (*i.e.* when $\boldsymbol{w} = \mathbf{0}$).

By considering the plane wave nature of presented problem, the solutions of Eqs. (10) and (11) may be taken under the form: $y(x_1, x_2) = \hat{y}(k_1, x_2)e^{\mathrm{i}k_1x_1}$. Noting that the operator $\mathbf{L}$ now becomes $\mathbf{L} = \mathrm{i}k_1\mathbf{L}_1 + \partial_2\mathbf{L}_2$, one has:

$$-\omega^2\mathbf{A}_1\mathbf{v} + k_1^2\mathbf{A}_2\mathbf{v} - \mathrm{i}k_1\mathbf{A}_3\partial_2\mathbf{v} - \partial_2\mathbb{t} = \mathbf{0}, \tag{17}$$

where

$$\mathbf{v} = \begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{w}} \end{pmatrix},\ \mathbb{t} = \begin{pmatrix} \mathbf{L}_2^T\hat{\boldsymbol{s}} \\ -\mathbf{L}_2^T\boldsymbol{m}\hat{p} \end{pmatrix} = ik_1\mathbf{A}_3^T\mathbf{v} + \mathbf{A}_4\partial_2\mathbf{v}, \tag{18}$$

and

$$\mathbf{A}_1 = \begin{bmatrix} \rho\mathbf{1} & \rho_f\mathbf{1} \\ \rho_f\mathbf{1} & \tilde{\boldsymbol{a}} \end{bmatrix},\quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{C}_u^{11} & \mathbf{C}_\alpha^{11} \\ (\mathbf{C}_\alpha^{11})^T & \mathbf{C}_M^{11} \end{bmatrix}, \tag{19a}$$

$$\mathbf{A}_3 = \begin{bmatrix} \mathbf{C}_u^{12} & \mathbf{C}_\alpha^{12} \\ (\mathbf{C}_\alpha^{21})^T & \mathbf{C}_M^{12} \end{bmatrix},\quad \mathbf{A}_4 = \begin{bmatrix} \mathbf{C}_u^{22} & \mathbf{C}_\alpha^{22} \\ (\mathbf{C}_\alpha^{22})^T & \mathbf{C}_M^{22} \end{bmatrix}. \tag{19b}$$

In the expressions of $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$, the 2-by-2 matrices $\mathbf{C}_u^{ab}, \mathbf{C}_\alpha^{ab}$ and $\mathbf{C}_M^{ab}$ with $a, b = 1, 2$ are defined by:

$$\mathbf{C}_u^{ab} = \mathbf{L}_a^T \mathbf{C}_u \mathbf{L}_b,\ \mathbf{C}_\alpha^{ab} = \mathbf{L}_a^T \mathbf{C}_\alpha \mathbf{L}_b,\ \mathbf{C}_M^{ab} = \mathbf{L}_a^T \mathbf{C}_M \mathbf{L}_b. \tag{20}$$

2.4 Boundary conditions at interfaces

At both interfaces Γ_1^{bf} and Γ_2^{bf}, the continuity of pressure and stress fields between the porous medium and the fluid domains requires. In addition, open-pore condition at the interfaces Γ_n^{bf} $(n = 1, 2)$ is assumed, requiring the continuity of normal fluid velocities. By writing the harmonic forms of the solutions ($p^{(n)} = \hat{p}^{(n)} \mathrm{e}^{\mathrm{i}k_1 x_1}$) and by noting that the normal unit vectors of Ω^b at two interfaces Γ_1^{bf} and Γ_2^{bf} (see Fig. 1) are defined by: $\boldsymbol{n}_1^{bf} = -\boldsymbol{n}_2^{bf} = \{0, 1\}^T$, the conditions at the interfaces may be expressed as follows:

$$\hat{u}_2 + \hat{w}_2 = \frac{1}{\rho_n \omega^2}\, \partial_2 \hat{p}^{(n)} \tag{21a}$$

$$\hat{p} = \hat{p}^{(n)} \tag{21b}$$

$$\hat{\boldsymbol{t}} = \{\ 0, -\hat{p}^{(n)}\}^T \tag{21c}$$

for $\forall \boldsymbol{x} \in \Gamma_n^{bf}$, $(n = 1, 2)$ where the traction vector $\hat{\boldsymbol{t}}$ is defined by $\hat{\boldsymbol{t}} = \mathbf{L}_2^T \hat{\boldsymbol{s}} = \{\hat{\sigma}_{21}, \hat{\sigma}_{22}\}^T$.

3 Finite element formulation

The weak formulation of the boundary value problem presented in (17) and (21a) may be now carried out using an usual procedure. Let $\mathscr{C}^{ad}$ be the set of admissible functions constituted by the sufficiently differentiable complex-valued functions. The conjugate transpose of $\delta\mathbf{v}$ is denoted by $\delta\mathbf{v}^*$. Upon integrating (17) against a test vector function $\delta\mathbf{v}$ and applying the Gauss theorem, then using the boundary condition (21c), the weak formulation of Eq. (17) reads:

$$\begin{aligned} &- \omega^2 \int_{-h}^{0} \delta\mathbf{v}^* \mathbf{A}_1 \mathbf{v}\ dx_2 + k_1^2 \int_{-h}^{0} \delta\mathbf{v}^* \mathbf{A}_2 \mathbf{v}\ dx_2 \\ &+ ik_1 \int_{-h}^{0} \left(\partial_2(\delta\mathbf{v}^*) \mathbf{A}_3^T \mathbf{v} - \delta\mathbf{v}^* \mathbf{A}_3 \partial_2 \mathbf{v}\right)\ dx_2 \\ &+ \int_{-h}^{0} \partial_2(\delta\mathbf{v}^*) \mathbf{A}_4 \partial_2 \mathbf{v}\ dx_2 +\ \delta\mathbf{v}^*(0) \mathbf{d}\ \hat{p}(0) \\ &- \delta\mathbf{v}^*(-h) \mathbf{d}\ \hat{p}(-h) = 0, \quad \forall\ \delta\mathbf{v} \in \mathscr{C}^{ad}, \end{aligned} \tag{22}$$

where $\mathbf{d} = \{0, 1, 0, 1\}^T$. In this weak formulation (22), the pore pressure $\hat{p}$ at $x_2 = 0$ and at $x_2 = -h$ are unknown variables but may be determined in terms of the displacement by using the conditions (21a)-(21b) and by taking into account the forms of general solutions in fluid domains presented in Section 2.2. First at the upper interface ($x_2 = 0$):

$$\omega^2 \left(\hat{u}_2 + \hat{w}_2\right) = \frac{ik_2^{(1)}}{\rho_1}(-P_I + P_R), \tag{23a}$$

$$\hat{p} = P_I + P_R, \tag{23b}$$

which lead to an impedance boundary condition:

$$\hat{p}(0) = \frac{\rho_1 \omega^2}{ik_2^{(1)}}(\hat{u}_2(0) + \hat{w}_2(0)) + 2P_I, \tag{24}$$

Similarly the impedance boundary condition at the lower interface ($x_2 = -h$) reads

$$\hat{p}(-h) = -\frac{\rho_2 \omega^2}{ik_2^{(2)}}(\hat{u}_2(-h) + \hat{w}_2(-h)). \tag{25}$$

By noting that $\hat{u}_2 + \hat{w}_2 = \mathbf{d}^T \mathbf{v}$ and by substituting Eqs. (24-25) into (22), we obtain

$$\begin{aligned} &- \omega^2 \int_{-h}^{0} \delta\mathbf{v}^* \mathbf{A}_1 \mathbf{v}\ dx_2 + k_1^2 \int_{-h}^{0} \delta\mathbf{v}^* \mathbf{A}_2 \mathbf{v}\ dx_2 \\ &+ ik_1 \int_{-h}^{0} \left(\partial_2(\delta\mathbf{v}^*) \mathbf{A}_3^T \mathbf{v} - \delta\mathbf{v}^* \mathbf{A}_3 \partial_2 \mathbf{v}\right)\ dx_2 \\ &+ \int_{-h}^{0} \partial_2(\delta\mathbf{v}^*) \mathbf{A}_4 \partial_2 \mathbf{v}\ dx_2 + \delta\mathbf{v}^*(0) \mathbf{D} \mathbf{v}(0) \\ &+ \delta\mathbf{v}^*(-h) \mathbf{D} \mathbf{v}(-h) = -2P_I \delta\mathbf{v}^*(0) \mathbf{d}, \\ &\forall \delta\mathbf{v} \in \mathscr{C}^{ad}, \end{aligned} \tag{26}$$

where $\mathbf{D} = \mathbf{d}\mathbf{d}^T$. We introduce a finite element discretization of the domain $[-h, 0]$ which contains n^{el} elements: $[-h, 0] = \bigcup_e \Omega_e$ with $e = 1, ..., n^{el}$. By using the standard Galerkin method, both functions $\mathbf{v}$ and $\delta\mathbf{v}$ in each element Ω_e are approximated using the same shape function:

$$\mathbf{v}(x_2) = \mathbf{N}_e \mathbf{V}_e, \quad \delta\mathbf{v}(x_2) = \mathbf{N}_e \delta\mathbf{V}_e, \quad \forall\, x_2 \in \Omega_e, \tag{27}$$

where $\mathbf{N}_e$ is the shape function, $\mathbf{V}_e$ and $\delta\mathbf{V}_e$ are the vectors of nodal solutions of $\mathbf{v}$ and $\delta\mathbf{v}$ within the element Ω_e, respectively. Substituting Eq. (27) into Eq. (26) and assembling the elementary matrices lead to a system of linear equations:

$$\left(\mathbf{K}^b + \mathbf{K}^\Gamma\right) \mathbf{V} = \mathbf{F}, \tag{28}$$

where $\mathbf{V}$ is the global nodal solution vector; $\mathbf{K}^b$ is the global "stiffness matrix" of the poroelastic layer; $\mathbf{K}_\Gamma$ represents the coupled operator between the fluid and poroelastic layers; the vector $\mathbf{F}$ is the external force vector due to the incident waves. Noting the number of nodes by N, the size of vectors $\mathbf{V}$ and $\mathbf{F}$ is $n^{eq} = 4N$ because each node has 4 degrees of freedom. The sizes of $\mathbf{K}^b$ and $\mathbf{K}^\Gamma$ are $n^{eq} \times n^{eq}$.

$$\mathbf{K}^b = -\omega^2\ \mathbf{K}_1 + k_1^2\ \mathbf{K}_2 + ik_1\ \mathbf{K}_3 + \mathbf{K}_4 \tag{29a}$$

$$\mathbf{K}_{jk}^\Gamma = \begin{cases} \frac{\rho_2 \omega^2}{ik_2^{(2)}} & \text{if } (j,k) = (2,2), (2,4), (4,2), (4,4) \\ \frac{\rho_1 \omega^2}{ik_2^{(1)}} & \text{if } (j,k) = (n^{eq}-2, n^{eq}-2), \\ & (n^{eq}-2, n^{eq}),\ (n^{eq}, n^{eq}-2),\ (n^{eq}, n^{eq}) \\ 0 & \text{otherwise} \end{cases} \tag{29b}$$

$$\mathbf{F}_j = \begin{cases} -2P_I & \text{if } j = n^{eq}-2, n^{eq} \\ 0 & \text{otherwise} \end{cases} \tag{29c}$$

where the matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$ are defined by:

$$\begin{aligned} \mathbf{K}_1 &= \bigcup_e \int_{\Omega_e} \mathbf{N}_e^T \mathbf{A}_1 \mathbf{N}_e\, dx_2, \\ \mathbf{K}_2 &= \bigcup_e \int_{\Omega_e} \mathbf{N}_e^T \mathbf{A}_2 \mathbf{N}_e\, dx_2 \\ \mathbf{K}_3 &= \bigcup_e \int_{\Omega_e} 2\left\{(\partial_2 \mathbf{N}_e)^T \mathbf{A}_3 \mathbf{N}_e\right\}_a\, dx_2, \\ \mathbf{K}_4 &= \bigcup_e \int_{\Omega_e} (\partial_2 \mathbf{N}_e)^T \mathbf{A}_4 (\partial_2 \mathbf{N}_e)\, dx_2, \end{aligned} \tag{30}$$

in which the notation $\{\star\}_a$ designates the anti-symmetric part of the $\{\star\}$.

After having solved the system of equations (28), the reflection (R) and transmission coefficients (T) can be computed:

$$R = \frac{\rho_1 \omega^2}{i k_2^{(1)}} \times \frac{\hat{u}_2(0) + \hat{w}_2(0)}{P_I} + 1, \tag{31a}$$

$$T = -\frac{\rho_2 \omega^2}{i k_2^{(2)}} \times \frac{\hat{u}_2(-h) + \hat{w}_2(-h)}{P_I}. \tag{31b}$$

4 Numerical examples

To validate the proposed formulations, we consider a homogeneous anisotropic poroelastic bone plate immersed in fluid. The interstitial fluid in porous medium is also assumed to be the same as the surrounding fluid domains, of which the mechanical properties are: $\rho_f = 1\ 000$ kg.m^{-3} et $K_f = 2.25$ GPa. The poroelastic material properties are given by: $\phi = 0.15$, $\rho_s = 1\ 722$ kg.m^{-3}, $c_{11}^u = 23.1$ GPa, $c_{22}^u = 1.51$ GPa, $c_{12}^u = 6.28$ GPa, $c_{66}^u = 4.8$ GPa, $c_{16}^u = c_{26}^u = 0$, $\alpha_{11} = 0.28$, $\alpha_{22} = 0.36$. The viscosity of interstitial fluid is $\eta = 1 \times 10^{-3}$ Pa.s and the permeability tensor is roughly taken by $\kappa_{11} = \kappa_{22} = 5 \times 10^{-13}$ m^2 The thickness of the bone plate is 4 mm.

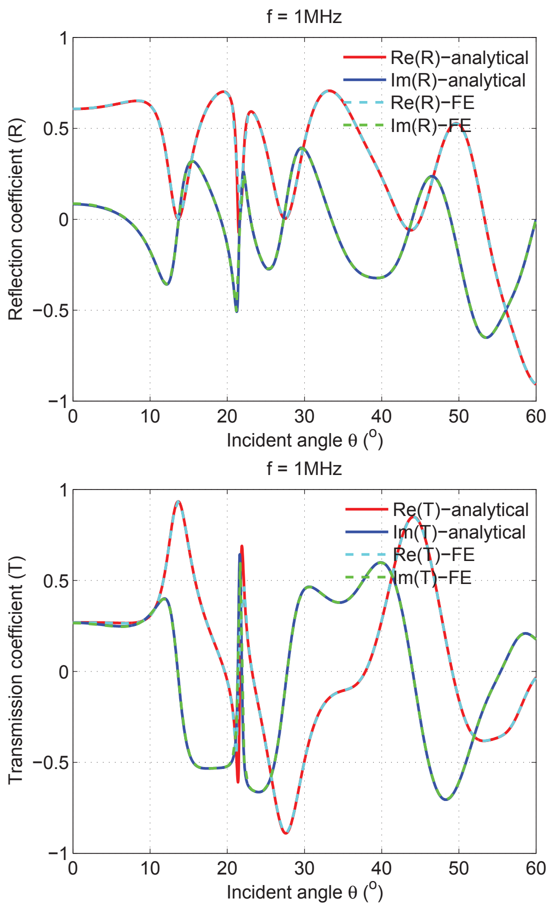

Figure 2: Reflection (R) and transmission (T) coefficients *versus* incident angle θ

Figure 2 presents the variation of the reflection and transmission coefficients with respect to the incident angle θ. The frequency of the incident wave is 1MHz. Both analytical and finite element solutions are presented. It may be seen that the FE solutions are perfectly matched with the analytical one. In order to get a better accuracy in high frequency, the spectral element method [7] have been employed been used. For this example, the bone plate has been discretized by only one 13 nodes element. The elementary matrices have been computed based upon the Gauss-Lobatto-Legendre integration rule.

5 Conclusion

A semi-analytical finite element formulation has been developed for analyzing acoustic response of a heterogeneous anisotropic poroelastic plate. The proposed method allows us to compute with high precision the reflection and transmission coefficients. Moreover, it has been shown that using the spectral elements with high-order interpolation functions give better results in comparing with the ones obtained by using the conventional finite elements. Moreover, it is straightforward to compute the dispersion relation (*ie* phase velocity and attenuation) of a heterogeneous anisotropic poroelastic plate by using developed formulations.

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