Modélisation de la diffusion acoustique par des particules solides : influence des forces de Faxen

T. Valier-Brasier[a], J.-M. Conoir[a], F. Coulouvrat[a] et J.-L. Thomas[b]
[a]Institut Jean le Rond d'Alembert, 4 place Jussieu, 75252 Paris, France
[b]Institut des Nanosciences de Paris, 4 place Jussieu, 75005 Paris, France
tony.valier-brasier@upmc.fr

Sound propagation in dilute suspensions is studied using two models, one based on the coupled phase equations which is a hydrodynamic model and the other based on the multiple scattering theory which is an acoustic model. Suspensions contain rigid or solid elastic particles which can be fixed or moving. A Rayleigh-Plesset like equation is obtained in order to take into account dilatational effects of elastic solid particles when using the hydrodynamic model. Analytical calculations performed for long wavelenght, low dilution and small weak bulk absorption in the ambient fluid show that both models are stricteley equivalent. It is shown that the Faxén forces introduced in the hydrodynamic model are implicitly taken into account in the acoustic model for fixed rigid and elastic solid particles while they are not for moving rigid particles. In this last case, the calculation of the acoustic scattering by a single particle has to be modified.

1 Introduction

The propagation of sound waves through dilute suspensions of different natures has been the subject of a large number of studies for many years since the pioneering article of Sewell [1] who considered immovable rigid particles suspended in a viscous atmosphere. In this context, two methods have been principally developed to study the influence of particles on the sound propagation in suspensions. First, the ECAH theory for which the wave number describing the sound propagation in suspensions is straightforwardly obtained from the multiple scattering theory based on the Foldy approximation [2]. The primary advantage of the ECAH theory is to be valid over the whole frequency range whatever the nature of the spherical particle, rigid, fluid or elastic. The second modelling is the coupled phase theory based on the two-phase hydrodynamic equations [3, 4] which gives a good framework to incorporate phenomena that would be difficult to include in the scattering theory such as non linear effects, mass transfers or chemical reactions. Moreover, after linearization, it also leads to an explicit dispersion equation that is generally simpler to calculate, which can be useful when dealing with the inverse problem. However, this theory is limited to the long wavelength regime and more complicated to applied to compressible particles other than bubbles for which many studies have been done [5].

The comparison of both models has already been widely discussed in the review of Challis *et al* [2]. However, most of the comparisons made in the literature are based on results of numerical calculations [3, 4] and it appears of fundamental interest to compare analytical calculations obtained by both models in order to make easier the physical interpretations. Of course, some assumptions are required to carry out analytical calculations. They are performed for spheres immersed in a viscous fluid in which thermal effects are neglected. The wavelength is assumed to be large compared to the size of particles and suspensions are supposed to be diluted enough.

A key point was to modify the equations of hydrodynamic models to take into account the compressibility of elastic solid spheres. Compared to Evans & Attenborough [4], we do not take into account thermal effects, but as for bubbles [5] we introduce a term of compressibility into the conservation of mass. This requires to find a relationship between the dynamic radius of elastic sheres and the acoustic pressure. To this end, we have established a new Rayleigh-Plesset-like equation under the hypothesis of long wavelength. The system of hydrodynamic equations is finally closed with the use of Faxén forces for rigid particles that involve Stokes viscous drag force, Basset-Boussinesq history force, added mass effect and Archimede force. As we do not exacly know Faxén forces for elastic solid particles we used those corresponding to rigid particles. We think that it is a realistic assumption when the compressibility of solid particles is not too high, which is confirmed later by our study.

In the long wavelength regime, it is well established that the propagation is governed by the two first modes of vibration of the particles. If A_n^L represent the amplitude of the compression wave field diffracted by a single particle, where n denotes the mode of vibration, A_0^L incorporates the effects of different compressibilities of the materials of the two phases (the particles and the surrounding viscous fluid). The mode $n = 0$ corresponds to a scattered radiation in a monopole form. The coefficient A_1^L represents viscous loss owing to the to-fro motion of the particle with respect to the surrounding fuid as well as scattered radiation in dipole form. In the long wavelength limit and when the density contrast between the phases is low, the solution tends to be dominated by A_0^L. When the density contrast increases, the A_1^L term tends to dominate and A_0^L can be neglected in many cases, and then the solution is essentially the hydrodynamic case. The goal of the paper is to analyze these behaviors from the analytical results which are obtained. For this, we have to consider two cases, heavy rigid particles (with infinite mass) which are fixed in space and free rigid particles (with a finite mass) which can move. These two cases are easily taken into account in the coupled phase theories by cancelling or not the velocity of particles in the Faxén forces. It is more complicated for the multiple scattering theory because Faxén forces never appear explicitly in this approach. However, the ECAH theory can be modified for rigid particles. We have just to replace the usual A_n^L coefficients associated to heavy rigid particles by those associated to moving rigid particles calculated by Temkin and Leung [6]. In this case we speak of modified ECAH theory. The elastic case is different. Surprisingly, we will see that there is no need to change the usual A_n^L coefficients calculated for elastic solid particles fixed in space. This suggests, at least in the long wavelengh regime, that Faxén forces and the moving of particles are implicitly taken into account in the multiple scattering theory when dealing with elastic solid particles.

2 Hydrodynamic model

As the hydrodynamic model is principally the one developed by Coulouvrat *et al* [7], we only give an outline of the general theory in the first subsection. The goal is to calculate the effective wave number k_H describing the sound propagation in suspensions. A Rayleigh-Plesset like equation is established in the second subsection in order to close the system of equations.

2.1 Effective wave number k_H

Spherical particles are assumed to be identical and randomly distributed within a viscous fluid which is characterized by the mass density ρ_0, the adiabatic sound speed c, the bulk η_v and shear η_s viscosities. In the following, p and v represent the mean pressure and velocity fields of the ambient fluid and v_p is the mean velocity field of the particles. The volume of the particles denoted $V_p = \frac{4}{3}\pi R^3$ depends on the dynamic radius R of the spheres. At equilibrum, we have $R = R_0$, $\rho = \rho_0$ and $V_p = V_{p0} = \frac{4}{3}\pi R_0^3$. The volume fraction of the particles within the total suspension volume is denoted as Φ_0 with $\Phi_0 = n_0 V_{p0}$ where n_0 is the number of particles per unit volume. In the following $m_p = \rho_p V_{p0}$ where m_p and ρ_p denote the mass and the density of particles respectively.

As we consider a linear acoustical behavior for waves of sufficiently small amplitude, the average field is governed by the following set of equations including the mass balance equation, the momentum balance equation and the second Newton's law [7]

$$\begin{cases} (1-\Phi_0)\left(\dfrac{\partial \rho}{\partial t} + \rho_0 \dfrac{\partial v}{\partial x}\right) = \rho_0 n_0 \dfrac{\partial V_p}{\partial t} - \rho_0 \Phi_0 \dfrac{\partial v_p}{\partial x}, \\ (1-\Phi_0)\rho_0 \dfrac{\partial v}{\partial t} + \dfrac{\partial p}{\partial x} - \left(\eta_v + \dfrac{4}{3}\eta_s\right)\dfrac{\partial^2 v}{\partial x^2} = -n_0 F_p, \\ m_p \dfrac{\partial v_p}{\partial t} = F_p, \end{cases} \quad (1)$$

where the Faxén forces are given by

$$F_p = 6\pi R_0 \eta_s (v - v_p) + 6R_0^2 \sqrt{\pi\rho\eta_s} \int_{-\infty}^{t} \frac{\partial(v - v_p)}{\partial t'} \frac{\mathrm{d}t'}{\sqrt{t-t'}} + \frac{2}{3}\pi\rho R_0^3 \frac{\partial(v - v_p)}{\partial t} + \frac{4}{3}\pi\rho R_0^3 \frac{\partial v}{\partial t}. \quad (2)$$

The compressibility is taken into account by the term $\rho_0 n_0 \partial V_p/\partial t$ in the conservation of mass. Assuming any component ψ of the different fields are plane waves which propagate in the direction parallel to the x axis, so that ψ is the sum of a static field and of an acoustic perturbation : $\psi = \psi_0 + \hat{\psi}\, exp[i(k_H x - \omega t)]$ with $\hat{\psi} << \psi_0$, the previous equations read

$$\begin{cases} (1-\Phi_0)\left(\dfrac{\omega}{c^2}\hat{p} - \rho_0 k_H \hat{v}\right) = \rho_0 \Phi_0 \left(3\omega \dfrac{\hat{R}}{R_0} + k_H \hat{v}_p\right), \\ -i\omega(1-\Phi_0)\rho_0 \hat{v} + i k_H \hat{p} + k_H^2 \left(\eta_v + \dfrac{4}{3}\eta_s\right)\hat{v} = -n_0 \hat{F}_p, \\ -i\omega m_p \hat{v}_p = \hat{F}_p, \\ \hat{F}_p = [A(\omega) + B(\omega)]\,\hat{v} - B(\omega)\hat{v}_p, \end{cases} \quad (3)$$

with

$$\begin{cases} A(\omega) = -i\dfrac{4}{3}\pi\rho\omega R_0^3, \\ B(\omega) = 6\pi\eta_s R_0 + 3(1-i)\pi\sqrt{2\rho\eta_s\omega}R_0^2 - i\dfrac{2}{3}\pi\rho\omega R_0^3, \end{cases} \quad (4)$$

where k_H denotes the effective wave number and ω the angular frequency. Combining the two last equations of the system Eq.(3) we get

$$\hat{v}_p = \frac{A+B}{Ar+B}\hat{v} \quad \text{and} \quad \hat{F}_p = \frac{(A+B)Ar}{Ar+B}\hat{v} \quad (5)$$

with $r = \rho_p/\rho_0$ the ratio of the particle density to the ambient fluid density.

In oder to close the system of equations, we have to intoduce a relation between the applied pressure in the ambient fluid $\hat{p}$ and the particle radius $\hat{R}$, the goal being to eliminate $\hat{R}$. As a general rule, Rayleigh-Plesset like equations, after linearisation, allow to find a function $C(\omega)$ such that

$$\hat{p} = C(\omega)\,\hat{R}. \quad (6)$$

Once $C(\omega)$ is known, it can be shown that the effective wave number k_H takes the following general form

$$\left(\frac{k_H}{k}\right)^2 = \frac{(1-\Phi_0)(1-D)(1+rT)}{1 + T - i(1-\Phi_0)(1-D)\omega\tau_v} \quad (7)$$

with $k = \omega/c$, and

$$\begin{cases} \tau_v = (\eta_v + \frac{4}{3}\eta_s)/\rho_0 c^2, \\ T(\omega) = \dfrac{\Phi_0}{1-\Phi_0}\dfrac{A(\omega)+B(\omega)}{A(\omega)r + B(\omega)} = \dfrac{\Phi_0}{1-\Phi_0}\,t(\omega), \\ D(\omega) = \dfrac{\Phi_0}{1-\Phi_0}\dfrac{3\rho_0 c^2}{R_0 C(\omega)} = \dfrac{\Phi_0}{1-\Phi_0}\,d(\omega). \end{cases} \quad (8)$$

In Eq.(8), τ_v is a characteristic time of viscosity, $d(\omega)$ represent dilatational effects due to the change of particle volume (radial oscillations described by the Rayleigh-Plesset-like equation) and $t(\omega)$ represent translational effects due to the viscous forces exerted on the particles (Faxén forces). If $\Phi_0 = 0$ we get $k_H = k_L$ with $k_L = k\,[1 - i\omega\tau_v]^{-1/2}$ which is as expected the usual dispersion equation in viscous fluids with no particles. In practical case of a dilute suspension ($\Phi_0 << 1$) with weak absorption ($\omega\tau_v << 1$), we have $(k_H/k)^2 - i\omega\tau_v \cong (k_H/k_L)^2$, and Eq.(7) becomes

$$\left(\frac{k_H}{k_L}\right)^2 = 1 - \Phi_0 - \Phi_0\, d(\omega) + (r-1)\Phi_0\, t(\omega). \quad (9)$$

For moving rigid speres without compressibility, $d(\omega) = 0$, and Eq.(9) simplifies to

$$\left(\frac{k_H}{k_L}\right)^2 = 1 - \Phi_0 + \Phi_0 \frac{(r-1)(A(\omega)+B(\omega))}{A(\omega)r + B(\omega)}. \quad (10)$$

If the sphere is assumed to be rigid and infinitely heavy ($r = \rho_p/\rho_0 \to \infty$) Eq.(10) reduces to

$$\left(\frac{k_H}{k_L}\right)^2 = 1 - \Phi_0 + \Phi_0\left(1 + \frac{B(\omega)}{A(\omega)}\right). \quad (11)$$

2.2 Rayleigh-Plesset like equation for elastic solid particles

The goal is here to express the dynamic radius of the elastic solid sphere as a function of the acoustic pressure at the surface of the sphere. Taking into account the weak compressibility and viscosity of the ambient fluid, the radial motion of the sphere in the ambient viscous fluid satisfy the Keller-Kolodner equation [8]

$$R\ddot{R}\left(1 - \frac{\dot{R}}{c}\right) + \frac{3}{2}\dot{R}^2\left(1 - \frac{\dot{R}}{3c}\right) = \left(1 + \frac{\dot{R}}{c}\right)H + \frac{R}{c}\dot{H}, \quad (12)$$

where the function H is given by

$$H(t) = \frac{1}{\rho}\left[p(R) - \tau_{rr}(R) - 4\eta_s\frac{\dot{R}}{R} - p_\infty\right]. \quad (13)$$

In the above equation, $p(R)$ and $\tau_{rr}(R)$ are the pressure and the normal viscous stress at the external surface of the sphere and p_∞ the pressure far from the sphere. Surface tensions ares neglected. Due to the hypothesis of long wavelengths, the radial displacement in the sphere u_r can be considered as quasi-static, so that

$$u_r(r,t) = A(t)r. \tag{14}$$

Let us note that this hypothesis will be ultimately validated by comparing the hydrodynamic model with the multiple scattering theory. The respect of the continuity of the velocity at the interface $r = R$ leads to write $\dot{A}(t) = \dot{R}(t)/R(t)$ and the integration of $\dot{A}(t)$ with respect to time yields

$$A(t) = \int_{-\infty}^{t} \frac{\dot{R}}{R}\mathrm{d}t = \ln\left(\frac{R}{R_0}\right). \tag{15}$$

As a consequence, the normal stress at the interface $r = R$ takes the following form

$$\sigma^p_{rr}(R) = 3K_p \ln\left(\frac{R}{R_0}\right), \tag{16}$$

where $K_p = \lambda_p + \frac{2}{3}\mu_p$ is the bulk modulus and λ_p and μ_p the Lamé parameters of the sphere. Neglecting surface tensions, the continuity of normal stresses at the interface are expressed by

$$\sigma^p_{rr}(R) = -p(R) + \tau_{rr}(R). \tag{17}$$

Inserting Eq.(16) and Eq.(17) into Eq.(13), we get

$$H(t) = -\frac{1}{\rho_0}\left[(3K_p)\ln\left(\frac{R}{R_0}\right) + 4\eta_s\frac{\dot{R}}{R} + p_\infty\right]. \tag{18}$$

The Keller-Kolodner equation Eq.(12) with $H(t)$ defined by Eq.(18) is what we call the Rayleigh-Plesset like equation for elastic solid particles.

The goal is now to linearize this one. Assuming that p_∞ is the sum of the static field p_0 and of an acoustic perturbation $\hat{p}$ with $\hat{p} << p_0$, we have $p_\infty(t) = p_0 + \hat{p}e^{-i\omega t}$ and the radius R can be written similarly $R(t) = R_e + \hat{R}e^{-i\omega t}$ where R_e is the radius of the sphere at the equilibrium state. Substituting the asymptotic expansions into the Rayleigh-Plesset like equation for elastic solid particles, and taking into account the continuity of normal stresses at the equilibrium state, namely

$$3K_p \ln\left(\frac{R_e}{R_0}\right) = -p_0, \tag{19}$$

provides

$$R_e(1 - i\overline{\omega})\,\hat{p} = (\rho c^2\overline{\omega}^2 + 4\omega\eta_s(i + \overline{\omega}) - 3K_p(1 - i\overline{\omega})\,)\hat{R} \tag{20}$$

with $\overline{\omega} = \omega R_e/c$. We finally obtain the looked for relation

$$\hat{p} = C(\omega)\hat{R} = \frac{\rho c^2\overline{\omega}^2 + 4\omega\eta_s(i + \overline{\omega}) - 3K_p(1 - i\overline{\omega})}{R_e(1 - i\overline{\omega})}\hat{R}. \tag{21}$$

At long wavelengths or low frequencies, $\overline{\omega}$ is very inferior to unity and $3K_p$ is superior to each term of the numerator in Eq.(21), it then follows that

$$\hat{p} = C(\omega)\hat{R} \cong -\frac{3K_p}{R_e}\hat{R}. \tag{22}$$

In such case, assuming that $R_e \cong R_0$, we have $d(\omega) \cong -\rho_0 c^2/K_p$ and Eq.(9) takes the form

$$\left(\frac{k_H}{k_L}\right)^2 = 1 + \Phi_0\left(1 - \frac{\rho c^2}{K_p}\right) + \Phi_0\frac{(r-1)(A+B)}{Ar+B}. \tag{23}$$

This is the basic equation which serves as reference in the following.

3 Acoustic model

According to the ECAH, the effective wavenumber k_A describing the sound propagation in suspensions is given by

$$\left(\frac{k_A}{k_L}\right)^2 = 1 - i\frac{3\Phi_0}{(k_L R_0)^3}\sum_{n=0}^{\infty}(2n+1)A^L_n \tag{24}$$

where k_L is the wave number of the longitudinal waves propagating in the ambient fluid and Φ_0 the concentration of spheres of radius R_0. Thus, only the amplitudes A^L_n of the longitudinal waves scattered in the ambient fluid have to be calculated. In the following, the center of the considered single sphere coincides with the origin of the coordinate system (O, r, θ, φ). As the incident longitudinal wave is assumed to propagate in the direction parallel to the x axis ($x = r\cos\theta$), the problem does not depend on the azymutal angle φ.

3.1 General equations

Without loss of generality and whathever the medium, viscous fluid or solid spheres, the displacement field $\mathbf{u}$ is the solution of the equation of motion

$$\rho\omega^2\mathbf{u} + (\lambda + 2\mu)\vec{\nabla}\vec{\nabla}.\mathbf{u} - \mu\vec{\nabla}\times\vec{\nabla}\times\mathbf{u} = \mathbf{0} \tag{25}$$

where ρ is the mass density and λ and μ are the Lamé parameters. Assuming the Helmholtz decomposition of the displacement $\mathbf{u} = \vec{\nabla}\phi_L + \vec{\nabla}\times(r\phi_T\mathbf{e}_r)$, the potentials ϕ_L and ϕ_T associated to the longitudinal and transverse components of the waves can be expanded in spherical harmonics ($m = L, T$)

$$\phi_m(r,\theta) = \sum_{n=0}^{\infty} i^n(2n+1)A^m_n z_n(k_m r)\,P_n(\cos\theta), \tag{26}$$

where $k_m = \omega/c_m$ are the wave numbers associated to the longitudinal $c_L = \sqrt{(\lambda+2\mu)/\rho}$ and shear $c_T = \sqrt{\mu/\rho}$ wave velocities, P_n are Legendre polynomials, z_n are spherical Bessel functions of order n and A^m_n are the unknown scattering coefficients calculated from the appropriate boundary conditions. The normal u_r and tangential u_θ displacements and the normal σ_{rr} and tangential $\sigma_{r\theta}$ stresses at the interface $r = R_0$ can therefore be expressed by

$$\begin{cases} u_r(R,\theta) = \displaystyle\sum_{n=0}^{\infty} i^n(2n+1)\frac{P_n(\cos\theta)}{R}\left(A^L_n U^L_n + A^T_n U^T_n\right), \\ u_\theta(R,\theta) = \displaystyle\sum_{n=0}^{\infty} i^n(2n+1)\frac{1}{R}\frac{\mathrm{d}P_n(\cos\theta)}{\mathrm{d}\theta}\left(A^L_n V^L_n + A^T_n V^T_n\right), \\ \sigma_{rr}(R,\theta) = \displaystyle\sum_{n=0}^{\infty} i^n(2n+1)\frac{P_n(\cos\theta)}{R^2}\left(A^L_n \Sigma^L_n + A^T_n \Sigma^T_n\right), \\ \sigma_{r\theta}(R,\theta) = \displaystyle\sum_{n=0}^{\infty} i^n(2n+1)\frac{1}{R^2}\frac{\mathrm{d}P_n(\cos\theta)}{\mathrm{d}\theta}\left(A^L_n T^L_n + A^T_n T^T_n\right), \end{cases} \tag{27}$$

where the functions U_n^m, V_n^m, Σ_n^m and T_n^m are given by

$$\begin{cases} U_n^L = nz_n(K_L) - K_L z_{n+1}(K_L), \\ U_n^T = n(n+1)z_n(K_T), \\ V_n^L = z_n(K_L), \\ V_n^T = (1+n)z_n(K_T) - K_T z_{n+1}(K_T), \\ \Sigma_n^L = \left[2n(n-1)\mu - (\lambda+2\mu)K_L^2\right] z_n(K_L), \\ \quad +4\mu K_L z_{n+1}(K_L), \\ \Sigma_n^T = 2n(n+1)\mu\left[(n-1)z_n(K_T) - K_T z_{n+1}(K_T)\right], \\ T_n^L = 2\mu\left[(n-1)z_n(K_L) - K_L z_{n+1}(K_L)\right], \\ T_n^T = \mu\left[\left(2(n^2-1) - K_T^2\right) z_n(K_T) \right. \\ \quad \left. +2K_T z_{n+1}(K_T)\right], \end{cases} \quad (28)$$

with $K_m = k_m R_0$ $(m = L, T)$.

3.2 Scattering coefficients for an elastic solid sphere fixed in space

From now on, the elastic solid sphere is characterized by the mass density ρ_p and the Lamé parameters λ_p and μ_p. The fluid is Newtonian and is characterized by the mass density ρ_0, the adiabatic sound speed c, the bulk η_v and shear η_s viscosities. In order to use the previous general expressions, the Lamé parameters of the fluid are defined by

$$\begin{cases} \lambda = \rho_0 c^2 - i\omega\left(\eta_v - \dfrac{2}{3}\eta_s\right), \\ \mu = -i\omega\eta_s, \end{cases} \quad (29)$$

so that $c_L = c\sqrt{1 - i\omega\tau_v}$ and $c_T = \dfrac{1-i}{2}\sqrt{\dfrac{2\omega\eta_s}{\rho_0}}$ where $\tau_v = (\eta_v + \frac{4}{3}\eta_s)/\rho_0 c^2$ is a characteristic time of viscosity.

In order to calculate the A_n^L scattering coefficients associated to the longitudinal waves scattered by an solid sphere fixed in space and immersed in a viscous fluid, the potentials must be specified for each wave. The scalar potential of the incident wave ϕ_I, those of scattered waves in the ambient fluid ϕ_m and those corresponding to the fields in the sphere ϕ_m^s $(m = L, T)$ are expressed by

$$\begin{cases} \phi_I(r,\theta) = \sum_{n=0}^{\infty} i^n(2n+1) j_n(k_L r) P_n(\cos\theta), \\ \phi_m(r,\theta) = \sum_{n=0}^{\infty} i^n(2n+1) A_n^m h_n(k_m r) P_n(\cos\theta), \\ \phi_m^s(r,\theta) = \sum_{n=0}^{\infty} i^n(2n+1) B_n^m j_n(k_m^s r) P_n(\cos\theta), \end{cases} \quad (30)$$

where j_n is the spherical Bessel function and h_n the spherical Hankel function of the first kind which is an outgoing spherical wave when considering the $exp(-i\omega t)$ time harmonic dependence. The amplitudes U_n^m, V_n^m, Σ_n^m and T_n^m $(m = I, L, T)$ of the waves in the fluid and the amplitudes $U_n^{(s,m)}$, $V_n^{(s,m)}$, $\Sigma_n^{(s,m)}$ and $T_n^{(s,m)}$ $(m = L, T)$ of the waves in the sphere are given by relations Eq.(28) with $K_m = k_m R$ and $K_m^s = k_m^s R$.

The boundary conditions for a solid sphere immersed in a viscous fluid are the continuity of normal and tangential displacements and the continuity of normal and tangential stresses. Substituting Eq. (27) and Eq.(28) into the boundary conditions leads to the matrix equation

$$\begin{pmatrix} U_n^L & U_n^T & -U_n^{p,L} & -U_n^{p,T} \\ V_n^L & V_n^T & -V_n^{p,L} & -V_n^{p,T} \\ \Sigma_n^L & \Sigma_n^T & -\Sigma_n^{p,L} & -\Sigma_n^{p,T} \\ T_n^L & T_n^T & -T_n^{p,L} & -T_n^{p,T} \end{pmatrix} \cdot \begin{pmatrix} A_n^L \\ A_n^T \\ B_n^L \\ B_n^T \end{pmatrix} = -\begin{pmatrix} U_n^I \\ V_n^I \\ \Sigma_n^I \\ T_n^I \end{pmatrix} \quad (31)$$

from which the amplitudes A_n^L are straightforwardly calculated. If the sphere is assumed to be infinitely rigid ($\lambda_p \to \infty$ and $\mu_p \to \infty$) and infinitely heavy ($\rho_p \to \infty$), the boundary conditions correspond to the cancellation of displacements u_r and u_θ of the fluid at the interface $r = R$. The amplitudes of the waves in the ambient fluid therefore satisfy the matrix equation

$$\begin{pmatrix} U_n^L & U_n^T \\ V_n^L & V_n^T \end{pmatrix} \cdot \begin{pmatrix} A_n^L \\ A_n^T \end{pmatrix} = -\begin{pmatrix} U_n^I \\ V_n^I \end{pmatrix}, \quad (32)$$

and takes the form

$$A_n^L = -\frac{f_n^I}{f_n^L}, \quad (33)$$

with

$$f_n^m = K_L z_{n+1}(K_L)\left[(n+1)h_n(K_T) - K_T h_{n+1}(K_T)\right] - nz_n(K_L) K_T h_{n+1}(K_T) \quad (34)$$

where $z_n \equiv j_n$ if $m = I$ and $z_n \equiv h_n$ if $m = L$.

3.3 Scattering coefficients for moving rigid spheres

The goal of this subsection is to take into account the mass $m_p = (4\pi R_0^3/3)\rho_p$ and the velocity $\mathbf{v}_p$ of a moving rigid sphere in the calculation of A_n^L. Unlike the previous case, Faxén forces are explicitly introduced to calculate the amplitudes A_n^L. This is the reason why we speak of modified ECAH theory when we use these coefficients that depend on the velocity $\mathbf{v}_p$ in order to obtain k_A. Analytical calculations follow broadly those of Temkin and Leung [6]. As the rigid sphere follows the polarization of the incident longitudinal plane wave, the velocity of the sphere is along the x-axis

$$\mathbf{v}_p = v_p e^{-i\omega t}\mathbf{e}_x. \quad (35)$$

This velocity is related to the linearized Faxén force $\mathbf{F}_p$ acting on the sphere by the relation

$$m_p \frac{\partial \mathbf{v}_p}{\partial t} = \mathbf{F}_p, \quad (36)$$

where $\mathbf{F}_p$ is given by

$$\mathbf{F}_p = \int_0^{2\pi}\int_0^{\pi} \overline{\overline{\sigma}}.\mathbf{e}_r R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \quad (37)$$

with $\overline{\overline{\sigma}}$ the stress tensor in the viscous fluid. Due to Eq.(36), the Faxén force $\mathbf{F}_p$ is also polarized along the x-axis ($\mathbf{F}_p = F_p.\mathbf{e}_x$). Inserting Eq.(27) into Eq.(37) yields

$$F_p = \frac{4}{3}\pi\left[A_1^L\left(\Sigma_1^L + 2T_1^L\right) + A_1^T\left(\Sigma_1^T + 2T_1^T\right) + \Sigma_1^I + 2T_1^I\right] \quad (38)$$

and using Eq.(36) we get

$$v_p = \frac{i}{\omega\rho_p R^3}\left[A_1^L\left(\Sigma_1^L + 2T_1^L\right) + A_1^T\left(\Sigma_1^T + 2T_1^T\right) + \Sigma_1^I + 2T_1^I\right]. \quad (39)$$

The amplitudes of scattered waves are then determined from the boundary conditions which are the continuity of normal and tangential velocities at the interface $r = R$, namely

$$\begin{cases} -i\omega\left(u_r + u_r^I\right) = v_p\cos\theta, \\ -i\omega\left(u_\theta + u_\theta^I\right) = -v_p\sin\theta. \end{cases} \quad (40)$$

Substituting Eq.(27) into the above boundary conditions gives

$$\begin{cases}\left[A_n^L U_n^L + A_n^T U_n^T + U_n^I\right] P_n(\cos\theta) = i\dfrac{R}{\omega} v_p \cos\theta, \\ \left[A_n^L V_n^L + A_n^T V_n^T + V_n^I\right] P_n^1(\cos\theta) = -i\dfrac{R}{\omega} v_p \sin\theta.\end{cases} \quad (41)$$

Due to the orthogonality of Legendre polynomials, the amplitudes of the modes $n \neq 1$ are solutions of the matrix equation Eq.(32) and that of the mode $n = 1$ is solution of the matrix equation

$$\begin{pmatrix} U_1^L + X_L & U_1^T + X_T \\ V_1^L + X_L & V_1^T + X_T \end{pmatrix} \cdot \begin{pmatrix} A_1^L \\ A_1^T \end{pmatrix} = -\begin{pmatrix} U_1^I + X_I \\ V_1^I + X_I \end{pmatrix}, \quad (42)$$

with

$$X_m = \frac{\Sigma_1^m + 2T_1^m}{\rho_p(\omega R)^2}, \qquad \text{where} \quad m = L, T, I. \quad (43)$$

Thus, the amplitudes of the modes $n \neq 1$ are given by Eq.(33) and that of the mode $n = 1$ by the following relation

$$A_1^L = -\frac{f_1^I}{f_1^L} \quad (44)$$

with

$$f_1^m = (U_1^m + X_m)(V_1^T + X_T) - (V_1^m + X_m)(U_1^T + X_T). \quad (45)$$

4 Analytical comparison of the two models

In order to compare both models analytically, it is necessary to make assumptions. They have already been introduced in section 2 where the dilution and the weak bulk absorption are supposed to be small. We have also implicitly assumed that K_p should be large enough so that the particles are not too compressible as in the case of bubbles. Finally, it is also assumed that wavelengths are large because it is a basic assumption of the hydrodynamic model. More accurately, we consider the frequency range defined by $kR_0 < 10^{-1}$. In this case, only the first two modes have to be taken into account in the ECAH theory.

The goal of this section is to get the asymptotic expansions of the wave numbers k_A of the acoustic model, and to compare these ones to those obtained in section 2 with the hydrodynamic model. We will successively compare three cases : the heavy rigid sphere, the moving rigid sphere and the elastic solid sphere.

Throughout the following, we use the asymptotic expansions of Bessel and Hankel spherical functions with small argument K_L in order simplify the analytic expressions. As the velocity of the bulk transverse wave in the fluid is much smaller than that of the bulk longitudinal wave, asymptotic expansions are used only for the spherical functions of variable K_L.

4.1 Heavy rigid spheres

Calculations are quite simple, we find that

$$\begin{cases} A_0^L \approx -i\dfrac{K_L^3}{3}, \\ A_1^L \approx -\dfrac{K_L^3}{6}\left(\dfrac{3i}{K_T^2} + \dfrac{3}{K_T} - i\right). \end{cases} \quad (46)$$

Taking into account Eq.(4), the amplitude A_1^L can be put in the form

$$A_1^L = i\frac{K_L^3}{9}\left(1 + \frac{B}{A}\right), \quad (47)$$

and inserting Eq.(46) into Eq.(24) yields

$$\left(\frac{k_A}{k_L}\right)^2 = 1 - \Phi_0 + \Phi_0\left(1 + \frac{B}{A}\right). \quad (48)$$

Therefore, the wavenumber calculated by both models are exactly the same at the order of approximation that we consider. As expected Faxén forces which are related to tanslational effects have an influence only on the amplitude of the mode $n = 1$.

4.2 Moving rigid spheres

The approximate expression of A_0^L is the same for fixed and moving rigid spheres, namely Eq.(46). It is not surprising because dilatational and translational effects are decoupled in dilute suspensions.

The calculation is more complicated for the mode $n = 1$. Inserting Eq.(28) with $n = 1$ in Eq.(45) gives

$$\begin{aligned} f_1^m =& K_T h_2(K_T)\left[K_L z_2(K_L) - z_1(K_L)\right] - 2K_L z_2(K_L) h_1(K_T) \\ &+ \frac{K_L K_T}{\rho_p(\omega R)^2}\left[2\mu K_T h_1(K_T) z_2(K_L)\right. \\ &\left. + (\lambda + 2\mu) K_L z_1(K_L) h_2(K_T)\right]. \end{aligned} \quad (49)$$

As before, taking into account exact expressions of $h_n(K_T)$ and asymptotic expansions of $j_n(K_L)$ and $h_n(K_L)$ and using the expression Eq.(4) of the functions $A(\omega)$ and $B(\omega)$, we get

$$f_1^I = \frac{iK_L e^{iK_T}}{6\pi R\eta_s r K_T^2}(A + B)(r - 1), \quad (50)$$

and

$$f_1^L = -\frac{3e^{iK_T}}{2\pi R\eta_s r(K_L K_T)^2}(rA + B). \quad (51)$$

Subsituting Eqs. (50) and Eq.(51) into Eq.(44) leads to the expression of the amplitude of the mode $n = 1$

$$A_1^L \approx i\frac{K_L^3}{9}\frac{(r-1)(A+B)}{Ar + B}. \quad (52)$$

Then, inserting Eq.(52) into Eq.(24) leads to the expression of the effective wavenumber

$$\left(\frac{k_A}{k_L}\right)^2 = 1 - \Phi_0 + \Phi_0\frac{(r-1)(A+B)}{Ar + B}. \quad (53)$$

For moving rigid spheres as for fixed rigid spheres, the effective wavenumber obtained by the acoustic model Eq.(53) and the hydrodynamic model Eq.(10) are exactly the same. Note that Faxén forces and the particle velocity ($\mathbf{v}_p \neq 0$) were explicitly introduced in both models for moving rigid spheres. So, we deal with the modified ECAH theory in this case.

4.3 Elastic solid spheres

After tedious calculations we can show that

$$A_0^L = -i\frac{K_L^3}{3}\frac{K_p - K}{K_p + \frac{4}{3}\mu} \quad (54)$$

with $K = \lambda + 2\mu/3$. If the bulk modulus K_p increases, the sphere tends to become rigid, and the amplitude of the mode $n = 0$ tends to the one related to fixed and moving rigid spheres Eq.(46). It may be noted that the mass of the sphere has no influence on the amplitude of the mode $n = 0$, that is to say on the dilatational effects. Moreover, it is worth noting that dilatational effects disappear if the bulk moduli of both media are equal ($K_p = K$). Moreover, we get

$$A_1^L = \frac{iK_L^3}{3}\frac{(1-r)(3-3iK_T-K_T^2)}{9iK_T-9+K_T^2+2r(K_T^F)^2}, \quad (55)$$

and taking into account the expressions Eq.(4) of functions $A(\omega)$ and $B(\omega)$, the amplitude of the mode $n = 1$ has finally for expression

$$A_1^L = \frac{iK_L^3}{9}\frac{(r-1)(A+B)}{B+rA}. \quad (56)$$

This one is exactly the same as for the moving rigid sphere Eq.(52). This means first that Faxén forces for elastic solid spheres are well approximated by Faxén forces of moving rigid spheres. Then, because the amplitudes A_1^L are the same, we can conclude that the velocity of the elastic solid sphere is taken into account by the ECAH theory even if the velocity ($\mathbf{v}_p \neq 0$) is not explicitely introduced in the calculation of elastic coefficients A_n^L as it was done for the moving rigid sphere. So it seems that the case of rigid spheres is apart, since we have to consider the case of fixed and moving rigid spheres separately.

Substituting the amplitude of the modes $n = 0$ and $n = 1$ into Eq.(24) leads to the expression of the effective wavenumber

$$\left(\frac{k_A}{k_L}\right)^2 = 1 - \Phi_0\frac{K_p - K}{K_p + \frac{4}{3}\mu} + \Phi_0\frac{(r-1)(A+B)}{B+rA}, \quad (57)$$

and, assuming that $K_p >> K$ with $K = \lambda + 2\mu = \rho_0 c^2 - 4\mu/3$, we get

$$\left(\frac{k_A}{k_L}\right)^2 = 1 - \Phi_0\left(1 - \frac{\rho_0 c^2}{K_p}\right) + \Phi_0\frac{(r-1)(A+B)}{B+rA}. \quad (58)$$

In the case of elastic spheres, the effective wavenumbers obtained by the acoustic Eq.(58) and hydrodynamic Eq.(23) models are exactly the same. Furthermore, if the bulk modulus K_p increases, the sphere tends to become rigid, and Eq.(58) tends to Eq.(53).

5 Conclusion

The comparison between the hydrodynamic and acoustic models describing the sound propagation in dilute suspensions of spheres has been performed for long wavelengths and small bulk absorption in the ambient fluid. On one side, the hydrodynamic model is a combination of Faxén formula, valid for a moving rigid sphere, and a Rayleigh-like equation to take into account the compressibility of the sphere. On the other side, the acoustic model is based on the long wave limit of ECAH theory for an elastic sphere embedded in a viscous fluid. Analytical calculations show that both models are strictly equivalent. Therefore, the Rayleigh-Plesset like model that we developed in order to describe the dilatation of elastic solid spheres and close the system of hydrodynamic equations is validated. The analytical calculations highlighted the following significant results :

⋆ Even if the center of the sphere is fixed in the acoutic model, the dipolar term, mode n=1, implies that the center of mass is moving. In linear regime, this contribution gives the Faxén formula exactly.

⋆ In linear regime, both contribution are independent. The mode $n = 0$ is related to the compressibility contrast between the sphere and the liquid. It models the differential change of volume between the two phases. On the contrary, the mode $n = 1$, does not depend on the compressibilities but takes into account the visco-inertial effects related to the contrast of density, i.e it models the relative displacement between the two phases.

⋆ If the sphere is assumed rigid in the ECAH model, the mode $n = 1$ is canceled by assumption. In that case, Faxén formula can be recovered only if the movement of the sphere is explicitly introduced.

If the comparison of both models has been performed for dilute suspensions of elastic solid spheres for which $K_p >> K$, a more general comparison for other particles like bubbles for which $K_p << K$ has to be done.

Remerciements

The authors acknowledge funding from the French Agengy for research (ANR-11BS09-007-03) project DiAMAN.

Références

[1] C.J.Sewell, On the extinction of sound in a viscous atmosphere by small obstacles of cylindrical and spherical form, *Philos.Trans.R.Soc.London* **210**, 239-270 (1910).

[2] R.E.Challis, M.J.W.Povey, M.L.Mather, A.K.Holmes, Ultrasound techniques for characterizing colloidal dispersions, *Rep. Prog. Phys.* **68**, 1541-1637 (2005).

[3] A.H.Harker, J.A.G.Temple, Velocity and attenuation of ultrasound in suspensions of particles in fluids, *J.Phys. D* **21**, 1576-1588 (1988).

[4] J.M.Evans, Coupled phase theory for soun propagation in emulsions, *J.Acoust.Soc.Am.* **102**, 278 (1997).

[5] K.W.Commander, A.Prosperetti, Linear pressure waves in bubbly liquids : Comparison between theory and experiments, *J.Acoust.Soc.Am.* **85**, 732-746 (1989).

[6] S.Temkin, C.M.Leung, On the velocity of a rigid sphere in a sound wave, *Journal of Sound and Vibration* **49**, 75-92 (1976).

[7] F.Coulouvrat, J.L.Thomas, K.Astafyeva, N.Taulier, J.M.Conoir, W.Urbach, A model for ultrasound absorption and dispersion in dilute suspensions of nanometric contrast agents, *J.Acoust.Soc.Am.* **132**, 3748-3759 (2012).

[8] J.B. Keller, I.I.Kolodner, Damping of underwater explosion bubble oscillations, *Journal of applied physics* **27**, 1152-1161 (1956).