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Approximate boundary conditions based on a complete transparent condition for the acoustic wave equation

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The numerical simulation of scattering problems generally involves particular boundary conditions set on the exterior boundary of the computational domain. These conditions are called Absorbing Boundary Conditions (ABC) when they satisfy the following properties; ABCs correspond to the approximation of a transparent condition, involve differential operators and minimize the reflections generated by the exterior boundary. Despite the works carried out on the design of ABCs, the existing ABCs need to be optimized and, recently, a new ABC has been proposed by Hagstrom et al. [8] . It is an improved Higdon ABC (IHABC) [9] where the amplitude of reflected waves is minimized by including a differential operator into the condition to model evanescent waves. The IHABC is very efficient when coupled with a finite element method, but it seems to hamper the Courant-Friedrichs-Lewy (CFL) condition when included into a Discontinuous Galerkin Method (DGM). Moreover, this condition is not easy to apply on arbitrarily-shaped boundaries. In this work, we address the issue of designing optimized ABCs which do not penalize the CFL condition when applying a DGM. We consider optimized ABCs adapted to arbitrarily-shaped regular boundaries and we construct a transparent condition based on the decomposition of the exact solution into a propagating field, an evanescent field and a grazing field. Then, a new condition is obtained by combining the approximations of the transparent condition in the three corresponding regions. It is not classical since it involves a fractional derivative arising from the grazing part of the solution. However, the condition is easily included into a finite element scheme and we have implemented it into an Interior Penalty Discontinuous Galerkin formulation. Numerical experiments have been performed and the results have shown that it does not modify the CFL condition. Furthermore, the absorption rate is improved when compared to classical ABCs.

1 Introduction

The truncation of the propagation domain is an important issue for the simulation of waves. It can be done by introducing a Perfectly Matched Layer condition (see [4] in which this idea is initiated and [10] for the application to acoustic waves) and this approach is now widely well-known to be used since it is very efficient. Nevertheless, in some cases such as scattering problems by elongated obstacles, the size of the computational box can be reduced significantly if the external boundary is adapted to the shape of the scatterer. Now the concept of PMLs seems to be more adapted to flat boundaries and, in case of curved surfaces, Absorbing Boundary Conditions (ABCs) might be more interesting. Most of ABCs are constructed from an approximation of the Dirichlet-to-Neumann operator and high-order conditions have been derived in order to improve the absorption power [8, 9]. Nevertheless, high-order conditions are generally written for piecewise-flat surfaces and corner conditions must also be introduced. In this work, we present a new ABC which is obtained from the combination of an ABC which construction and performance are described in [3] and a boundary condition including the behavior of grazing waves near the external surface of the computational domain.

2 General setting

In this paper, we consider the time-dependent wave equation in a two-dimensional domain Ω with a sound-hard obstacle inside and an ABC on its external boundary. We have

$$(\mathcal{S})\begin{cases}\partial_t^2 u - \operatorname{div}\left(c^2\nabla u\right) = f, & \text{in } (0,T)\times\Omega,\\ u(0,x)=0\ ;\ \partial_t u(0,x)=0, & \text{in } \Omega,\\ \partial_{\mathbf{n}} u = 0, & \text{on } \Gamma_N,\\ \partial_{\mathbf{n}} u = Bu, & \text{on } \Gamma_{\text{abs}},\end{cases} \tag{1}$$

where f is the source function, c the velocity of the wave u (the unknown field), T the final time, $\mathbf{n}$ the unit outward normal vector, Γ_N and Γ_{abs} respectively the boundary of the obstacle and the ABC which is represented by the operator B. The operator B can be differential, for instance, it reads $-\frac{1}{c}\partial_t - \frac{\kappa}{2}$ which corresponds to the curvature ABC (C-ABC) which is a well-known ABC that only takes into account propagating waves but it can also be a pseudo-differential operator.
In this paper, we set $c = 1$ for a sake of simplicity and we restrict our study to the 2D case but the extension to the 3D case is relatively straight forward.
In [3], we have introduced a new ABC which is defined by :

Theorem 2.1 *A second-order family of ABCs taking both propagating and evanescent waves into account reads as*

$$(\partial_{\boldsymbol{n}}+\sigma)\left(\partial_t+\partial_{\boldsymbol{n}}+\frac{\kappa}{2}\right)u=0 \text{ on } \Gamma_{abs}. \tag{2}$$

The construction of this condition is based on the micro-diagonalization [13] of the acoustic wave equation both in the hyperbolic and the elliptic region.We assume that the neighborhood of Γ_{abs} can be parametrized by (r, s) in such a way that $\Gamma_{\text{abs}}=\{(r, s)=(0, s)\}$. Then, r stands for the radial distance to Γ_{abs} and s denotes the curvilinear abscissa. We then consider the solution to the acoustic wave equation in the Fourier domain (ω and ξ denote respectively the dual variables of t and s) represented by the elliptic frequencies $\{(\omega,\xi), h^{-2}\xi^2-\omega^2>0\}$ and the hyperbolic frequencies $\{(\omega,\xi), h^{-2}\xi^2-\omega^2<0\}$ which respectively correspond to the evanescent and the propagating waves. We have used the following notations : κ is the curvature of the external boundary Γ_{abs}, $h=1+r\kappa$ is a parameter depending on the radial variable r while κ depends on s.
We can observed that the glancing region $\{(\omega,\xi), h^{-2}\xi^2-\omega^2=0\}$ which corresponds to the grazing waves is not considered. This is due to the fact that the micro-diagonalization technique can not be applied to these frequencies. As a consequence, the resulting boundary condition does not take the corresponding waves into account. In this work, we concentrate on the modelling of the complete wave field and we propose a new ABC including the glancing region which performance is illustrated by some numerical experiments.

3 The new ABC

We cannot apply the micro-diagonalization in the region $\{(\omega,\xi), h^{-2}\xi^2-\omega^2=0\}$ since in that region, the system does not admit eigenvalues. We thus perform an asymptotic expansion to design an ABC that takes the grazing waves into account [5]. We first apply a partial Fourier transform in the variable t to the acoustic wave equation and we consider its principal symbol in the neighborhood of Γ_{abs}:

$$\omega^2u+\partial_r^2u+h^{-2}\partial_s^2u=0. \tag{3}$$

In the neighborhood of Γ_{abs}, the radial distance satisfies $r<<1$. Then thanks to a Taylor expansion we obtain

$$h^{-2}\sim 1-2r\kappa. \tag{4}$$

We now apply a Fourier transform on the variable s and we get that the Fourier transform of u denoted by $\hat{u}$ satisfies

$$\partial_r^2\widehat{u}+\left[\omega^2-\xi^2\left(1-2r\kappa\right)\right]\widehat{u}=0, \tag{5}$$

which is an Airy equation (see [1]). Hence, $\hat{u}$ can be written as

$$\widehat{u}=A\,Ai(\alpha r+\beta)+B\,Bi(\alpha r+\beta), \tag{6}$$

where A, B, α and β are constant with respect to r, Ai and Bi are the Airy functions [1]. Since we are considering outgoing wave fields, the Airy function Bi must be removed and we thus have $B=0$. Then using the properties of Ai (see [1] eq. (10.4.1) p.446) and (5), we get

$$\widehat{u}=A\,Ai\left((i\xi)^{2/3}(2\kappa)^{1/3}\left(r+\frac{\omega^2-\xi^2}{2\xi^2\kappa}\right)\right). \tag{7}$$

Finally, the grazing waves are represented by (7) and we can then evaluate $\partial_r\widehat{u}_{|r=0}=\partial_{\mathbf{n}}\widehat{u}_{|r=0}$. Since we consider the region where $\omega^2=h^{-2}\xi^2$ and knowing that $h=1$ on Γ_{abs}, we then have

$$\partial_r\widehat{u}_{|r=0}=(i\xi)^{2/3}(2\kappa)^{1/3}\frac{Ai'(0)}{Ai(0)}\widehat{u}_{|r=0}. \tag{8}$$

To obtain a condition written for u, we have to apply an inverse Fourier which requires to give a sense to the inverse Fourier transform of $(i\xi)^{2/3}\,\hat{u}$ which leads to deal with a fractional derivative [11].

Definition The Caputo fractional derivative of $D_X^a g$ of order a ($a\in\mathbb{R}$) of a given function g is defined by

$$\mathcal{F}\left(D_X^a g(X)\right)=(ik_X)^a\,\mathcal{F}(g(X)), \tag{9}$$

where $\mathcal{F}$ denotes the Fourier transform and k_X the dual variable of X.

According to [1] (eq. (10.4.5) p.446),

$$Ai'(0)=-\frac{1}{3^{1/3}\Gamma(1/3)}, \tag{10}$$

and (eq. (10.4.4) p.446)

$$Ai(0)=\frac{1}{3^{2/3}\Gamma(2/3)}, \tag{11}$$

therefore, after applying an inverse Fourier transform on (8), we get

Theorem 3.1 *Grazing waves can be represented on* Γ_{abs} *by*

$$\partial_{\boldsymbol{n}}u+(6\kappa)^{1/3}\frac{\Gamma(2/3)}{\Gamma(1/3)}D_s^{2/3}u=0 \text{ on } \Gamma_{abs}. \tag{12}$$

To obtain a condition that both takes into account the waves corresponding to the hyperbolic and the elliptic regions and to the interface between these two regions, we compose (12) with (2).

Theorem 3.2 *A third-order family of ABCs taking propagating, evanescent and grazing waves into account reads as*

$$\left(\partial_{\boldsymbol{n}}+c_rD_s^{2/3}\right)(\partial_{\boldsymbol{n}}+\sigma)\left(\partial_t+\partial_{\boldsymbol{n}}+\frac{\kappa}{2}\right)u=0 \text{ on } \Gamma_{abs}, \tag{13}$$

with $c_r:=(6\kappa)^{1/3}\frac{\Gamma(2/3)}{\Gamma(1/3)}$.

Remark We could have only considered the case of an ABC that only takes both propagating and grazing waves into account. This ABC is given by

$$\left(\partial_{\mathbf{n}}+c_rD_s^{2/3}\right)\left(\partial_t+\partial_{\mathbf{n}}+\frac{\kappa}{2}\right)u=0 \text{ on } \Gamma_{\text{abs}}. \tag{14}$$

4 Numerical experiments

Here, we only present the results obtained with (14), the implementation of (13) is still in progress.
In [3], to incorporate the ABC (2) into a finite element formulation, we have proposed to rewrite it in a more convenient way which can be easily included in a variational formulation. After a space and time discretization, we have tested the ABC in two given configurations and we have numerically observed that there exists an optimal choice of σ. Moreover, taking into account evanescent waves seems to be a good idea to improve the efficiency of the solution.
As in [3], to implement the ABC (14) into a finite element formulation, we propose to use a more convenient expression of (14) which can be easily introduced into a variational formulation thanks to an auxiliary unknown ψ. The ABC (14) is rewritten on Γ_{abs} as

$$\partial_{\mathbf{n}} u = -\partial_t u - \frac{\kappa}{2} u + \left(\partial_t - \frac{\kappa}{2} + c_r D_s^{2/3}\right)^{-1} \left(\partial_s^2 - \frac{\kappa^2}{4}\right) u$$

and we define ψ as the surface field satisfying

$$\left(c_r D_s^{2/3} + \partial_t - \frac{\kappa}{2}\right)\psi = \left(\partial_s^2 - \frac{\kappa^2}{4}\right) u \text{ on } \Gamma_{\text{abs}}.$$

Then the solution u satisfies

$$\partial_{\mathbf{n}} u = -\partial_t u - \frac{\kappa}{2} u + \psi \text{ on } \Gamma_{\text{abs}}.$$

To deal with the fractional derivative term, we set another auxiliary variable Φ only defined on Γ_{abs} by $\Phi = D_s^{2/3}\psi$.
For the space discretization, we consider an Interior Penalty Discontinuous Galerkin (IPDG) method [2, 7] and a second-order Leap-Frog scheme for the time discretization which is quasi-explicit since all the matrices are block-diagonal and therefore easily invertible. To evaluate Φ at each time step, we use a finite difference scheme on Γ_{abs} which is known as the Shifted Grünwald formula [12]

$$\Phi(s_i, n\Delta t) = \frac{1}{h^{2/3}} \sum_{j=0}^{i+1} g_j \psi(s_{i-j+1}, n\Delta t), \tag{15}$$

where $s_i = ih$, h is the space step on Γ_{abs} (supposed to be constant), Δt the time step and the weights g_j are given by $g_0 = 1$, $g_1 = -\frac{2}{3}$ and

$$g_j = -\frac{g_{j-1}}{j}\left(\frac{5}{3} - j\right) < g_{j-1}.$$

This scheme seems to be global on Γ_{abs} but it is in fact pseudo-local. Indeed, at each time step, to find the value of Φ at a given point s_i we don't have to consider the value of ψ at all the previous points but only at the points in the neighborhood of s_i since the weights g_j corresponding to points s_{i-j+1} far from s_i are very small.
We have compared the performances of the ABC (14) to the ones of the C-ABC and the ABC (2) for two simple configurations.
In the first configuration, the domain Ω is a disk of radius 3m, centered in $(0,0)$ and Γ_{abs} is the boundary of Ω. We consider zero initial condition and an off-center point source in space at $(0, 1\text{m})$ which is a second-derivative of a Gaussian with a dominant frequency of $1Hz$.
In the second configuration, the domain Ω is a ring centered in $(0,0)$ of internal radius 1m and of external radius 3m. Γ_{abs} is the external boundary of the ring and Γ_N is the internal one. We consider zero initial conditions and an off-center point source in space at $(0, 1.5\text{m})$ which is a second-derivative of a Gaussian with a dominant frequency of $1Hz$.
In both configurations, we set the final time $T = 40s$.
To compare the efficiency of the different ABCs, we set three receivers near the absorbing boundary at points $(0, 2.85\text{m})$, $(-2\text{m}, 2\text{m})$ and $(-2.85\text{m}, 0)$ and we compute the relative $L^2_{(x,y)}([0,T])$ error at each receiver which coordinates are (x,y). This error is defined by

$$\frac{\left(\int_0^T (u_{app}(t,(x,y)) - u_{ex}(t,(x,y)))^2 dt\right)^{1/2}}{\left(\int_0^T (u_{ex}(t,(x,y)))^2 dt\right)^{1/2}}$$

where u_{app} is the approximation of the solution and u_{ex} is the exact solution obtained thanks to a Cagniard-de Hoop method [6]. The error is given after 6000 iterations (with a time step equal to $7e-3s$). According to [3], in those given configurations the optimal choice of σ seems to be $\sigma = 0.7$, that's why we compare the new ABC to the ABC (2) with $\sigma = 0.7$. In Tab.1, we give the results for the first configuration and in Tab.2, the ones for the second one. We can observe that the solution at the first receiver located above the source is very accurate, whatever the condition is. This is due to the fact that most of the waves impinge the boundary at normal incidence above the source. On the contrary, the solutions obtained at the two other receivers are more accurate with the new condition than with the C-ABC or the ABC (2).

	(0,2.85)	(-2,2)	(-2.85,0)
C-ABC	1.14	2.35	3.15
prop-ev ($\sigma = 0.7$)	1.01	1.90	2.59
prop-grazing	0.97	1.68	2.40

Table 1: Relative L^2 error (in %) - circle

Moreover, we have computed the solution with the new ABC until $T = 130s$ (20000 iterations) and we observe that the solution remained stable. We thus claim that the new condition preserves the long-time stability of the wave equation.
Nevertheless we must observe that the experiments we have carried out concern the case of a circular boundary, for which the curvature ABC is already very accurate. Therefore we are now investigating the performances of the new ABC on an elliptic boundary. In particular, we wish to know if it can improve significantly the accuracy of the solution obtained with the C-ABC or with ABC (2).

	(0,2.85)	(-2,2)	(-2.85,0)
C-ABC	1.42	7.21	5.77
prop-ev ($\sigma = 0.7$)	1.22	6.03	4.84
prop-grazing	1.11	5.46	4.44

Table 2: Relative L^2 error (in %) - ring

5 Conclusion

In this paper, we have proposed a new ABC for the acoustic wave equation that can be justify for any arbitrarily shaped surface by using a micro-local diagonalization process or an asymptotic expansion. This ABC has been written for all regular convex domains and consider the hyperbolic, the elliptic regions and the frontier between these two regions. We have performed some numerical tests in simple configurations and we have observed that the accuracy of the solution is improved. We plan to test numerically other configurations as elliptic domains or more general convex domains. We expect to see more significant improvements in the accuracy of the solution with such configurations than in the case of circular domains. We also have to implement the complete ABC (13) and we expect the solution to be more accurate. Even if the construction of the ABC in the 3D case is the same as in the 2D case, the implementation of the ABCs involving a fractional derivative are not easy. Indeed, to the best of our knowledge, there is no existing schemes for such a configuration. To overcome this difficulty, we plan to consider the fractional derivative with respect to t instead of the fractional derivative with respect to s, since in the case of the grazing waves $\omega^2 = \xi^2$. In that case, there would not be any difference between the 2D case and the 3D case.

References

[1] M. Abramovitz, I. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover Publications, New York, 1972.

[2] M. Ainsworth, P. Monk, W. Muniz, Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second-order wave equation, J. Sci. Comp., 27 (1-3), 2006.

[3] H. Barucq, J. Diaz, V. Duprat, Micro-differential boundary conditions modelling the absorption of acoustic waves by 2D arbitrarily-shaped convex surfaces, Comm. in Comp. Phys., submitted.

[4] J.P. Berenger, A perfectly matched layer for the absorption of electromagnetic waves, J. Comp. Phys. 114 (1994), 185-200.

[5] D. Bouche, F. Molinet, Méthodes asymptotiques en électromagnétisme, Springer-Verlag, 1994.

[6] L. Cagniard, Reflection and refraction of progressive seismic waves, McGraw-Hill, 1962.

[7] M.J. Grote, A. Schneebeli, D. Schötzau, Discontinuous Galerkin finite element method for the wave equation, SIAM J. Num. Anal., 44 (2006), 2408-2431.

[8] T. Hagstrom, A. Mar-Or, D. Givoli, High-order local absorbing conditions for the wave equation : extensions and improvements, J. Comp. Phys., 227 (2008), 3322-3357.

[9] R. Higdon, Numerical absorbing boundary conditions for the wave equation, Math. Comp., 49 (1987), 65-90.

[10] D. Komatitsch, J. Tromp, A perfectly matched layer absorbing boundary condition for the second-order seismic wave equation, Geophys. J. Int. (2003) 154, 146-153.

[11] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[12] S.G. Samko,A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Yverdon, Switzerland, Gordon and Breach, 1993.

[13] M.E. Taylor, Pseudodifferential Operators, Princeton University Press, NJ, 1981.